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EFFICIENCY EVALUATION OF WEIBULL DISTRIBUTION PARAMETER ESTIMATORS FOR FAILURE DATA: A COMPARATIVE STUDY USING SIMULATION EXPERIMENTS

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Abstract
The purpose of this study is to assess the efficiency of parameter estimation methods of Weibull distribution for the analysis of failure data and reliability data. This is done by comparing three approaches for estimation: Maximum Likelihood Estimation (MLE), Method of Moments (MoM) and Least Squares Method (LSM). The significance of this study lies in the fact that the Weibull distribution is widely used to model failure times because of its high flexibility and it can be used to model many failure rate patterns.

The method that is used as a simulation method is carried out by applying software called Stata 17 with the programming language Mata. The two-parameter Weibull distribution was used to generate data and had different values for the shape parameter (β) and sample sizes. The performance of the estimation methods was then assessed with the help of two statistical criteria: Mean Squared Error (MSE) and Bias.

The results indicate that the Mean Squared Error (MSE) and the Bias values of the Maximum Likelihood Estimation (MLE) method were smaller than the other methods in most of the cases. This means that it has a high efficiency for estimating the parameters of the Weibull distribution, especially with larger sample sizes. The results also showed that the Least Squares Method was good and close to the Maximum Likelihood Estimation method so that it can be used as an alternative method in some practical situations. The Method of Moments was found to be the least efficient method, particularly for high values of the shape parameter or for small sample sizes.

This study concludes that the choice of estimation method is mainly dependent on the sample size and failure data. It also suggests that in practical applications of reliability analysis and survival data the Maximum Likelihood Estimation method should be adopted as a standard method.

1. Introduction

The analysis of lifetime data is regarded to be one of the important branches of applied statistics as it is a fundamental ingredient in the study of system lifetimes and assessment of their performance over time (Gross & Clark, 1975). It is important to select the appropriate probability distribution as it plays a critical role in the model of failure time that is used, and thus influences the quality of the estimation and statistical inference. The purpose of reliability studies is to provide a description of the failure behavior and prediction of the lifetime of different components and systems in various applications. The accuracy of such studies is mostly dependent on the choice of a probabilistic model that is flexible and can accurately model the nature of real data.

The Weibull distribution is a very versatile continuous probability distribution and can also model various hazard rate patterns, including early failure, random failure and aging failure (Abernethy, 2006); therefore, it has a special place in failure data analysis. This has contributed to its use as one of the most popular distributions in reliability modeling and survival analysis. This distribution is important because it is dependent on two parameters, the shape parameter and the scale parameter, that determine its properties and behavior.

The accuracy and efficiency of parameter estimation are important factors in the efficiency of the use of the Weibull distribution. There are many different methods for statistical estimation, including Maximum Likelihood Estimation, Method of Moments, and the Least Squares Method, with varying statistical characteristics in bias, variance, and efficiency. So it is important to have comparative studies to assess their performance and determine which method is most effective for various conditions (Jeffrey, 2003).

This study aims to compare the efficiency of Maximum Likelihood Estimation, Method of Moments and Least Squares Method in estimating the parameters of Weibull distribution, namely the shape parameter (β) and scale parameter (α). These methods perform differently for different sample sizes and parameter values, thus it is suitable to use a simulation approach to compare their statistical efficiency by Mean Squared Error (MSE) and Bias. The purpose of the study is to determine the best estimation technique for real-life problems where reliability and lifetime data are applied.

Weibull Distribution

The Weibull distribution (Nelson, 2004) is a continuous failure-time distribution and is one of the most widely used distributions in failure data analysis and reliability studies. It was named after the Swedish scientist Waloddi Weibull, who introduced it in 1939. This distribution is characterized by its high flexibility when applied to various types of data, and it has extensive applications in life data analysis.

The Weibull distribution can be derived from the concept of the hazard rate, which serves as a fundamental basis for describing failure behavior and determining the form of the distribution. The hazard function of the Weibull distribution is given by (Meeker & Escobar, 1998):

$$h(t) = \frac{\alpha}{\beta} t^{\alpha-1} \qquad (1)$$

Where: A random variable representing the time of failure occurrence

$\beta > 0$: represents the scale parameter.

$\alpha > 0$: represents the shape parameter.

And by applying the following relationship between the distribution function and the hazard function:

$$f_t(t) = h(t) \cdot e^{-\int_0^t h(u)\,du}$$

We obtain the two-parameter probability density function, which is:

$$f_t(t) = \begin{cases} \frac{\alpha}{\beta} t^{\alpha-1} exp\left(-\frac{t^{\alpha}}{\beta}\right) & t > 0 \\ 0 & O.W \end{cases} \qquad (2)$$

The cumulative distribution function can be obtained, which is:

$$F_t(t) = P(T < t) = \int_0^t f(u)\,du = 1 - exp\left(-\frac{t^{\alpha}}{\beta}\right) \qquad (3)$$

It is worth noting that :

(1) The hazard function is a decreasing function with respect to time t when $\alpha < 1$.

(2) The hazard function is an increasing function with respect to time t when $\alpha > 1$.

(3) The hazard function is constant when $\alpha = 1$.

The survival function for this distribution is:

$$S(t) = e^{-\beta t^{\alpha}} \qquad (4)$$

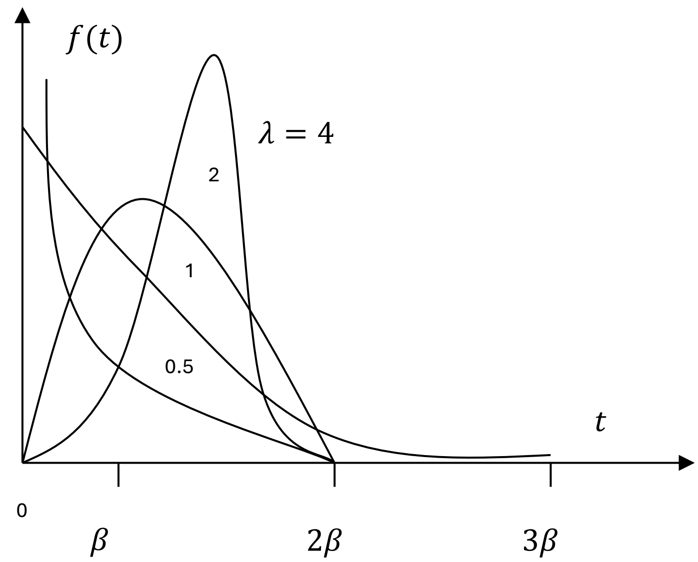
And the mean, variance, and skewness coefficient of the Weibull distribution are given by the following expressions:

$$Mean = \beta^{\frac{1}{\alpha}}\Gamma(1 + 1/\alpha)$$

$$Variance = \beta^{\frac{2}{\alpha}}\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$$

$$Skewness = \frac{\Gamma\left(1+\frac{3}{\alpha}\right) - 3\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right) + 2\Gamma^3\left(1+\frac{1}{\alpha}\right)}{\left[\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{3}{2}}}$$

The probability density function of the Weibull distribution can be represented graphically (Nelson, 1982):



Weibull probability densities

Figure 1. Different shapes of the probability density function of the Weibull distribution.

2. Method

Traditional Estimation Methods

Traditional methods for estimating the parameters of the Weibull distribution are based on the assumption that the parameter to be estimated is fixed, and estimation is performed using only the observed sample data. The most important of these methods are:

2.1. Maximum Likelihood Method (ML) (Wayne & Rootunmala, 1986)

This method possesses several desirable properties and is based on estimating the parameter values that maximize the likelihood function. The likelihood function can be defined as follows:

If t1, t2, ….. , tn represent the elements of a random sample of size (n) drawn from a population with a known probability density function $f(t, \theta)$, then the likelihood function, denoted by (L) , is the joint probability function, that is:

$$L = f(t_1, \theta).f(t_2, \theta) \dots f(t_n, \theta) = \prod_{i=1}^{n} f(t_1, \theta) \quad (5)$$

And the likelihood function of the Weibull distribution is:

$$L(t_1,t_2,\dots,t_n,\alpha,\beta) \;=\; \prod_{i=1}^{n} \frac{\alpha}{\beta}\; t_i^{\alpha-1}\; e^{-\frac{t_i^{\alpha}}{\beta}} \qquad (6)$$
$$= \frac{\alpha^n}{\beta^n}\; e^{-\frac{\sum_{i=1}^{n} t_i^{\alpha}}{\beta}} \prod_{i=1}^{n} t_i^{\alpha-1}$$

And in order to estimate the likelihood function, it must be transformed into a linear form by taking the natural logarithm of both sides of equation (6), we obtain:

$$\ln L \;=\; n \ln\alpha \;-n\ln\beta \;-\frac{\sum_{i=1}^{n} t_i^{\alpha}}{\beta} +\; (\alpha-1)\sum_{i=1}^{n} \ln t_i \qquad (7)$$

The maxima of function (7) are computed to obtain the estimated values of both the shape parameter and the scale parameter (β,α) , which maximize the likelihood function, as follows:

We find the partial derivative of the function with respect to the parameters(β,α) , and by equating the partial derivatives to zero, we obtain :

$$\frac{\partial \ln L}{\partial\alpha} \;= \frac{n}{\hat{\alpha}} \;-\; \frac{\sum_{i=1}^{n} t_i^{\hat{\alpha}} \ln t_i}{\hat{\beta}} \;+\; \sum_{i=1}^{n} \ln t_i \;=\; 0 \qquad (8)$$

$$\frac{\partial \ln L}{\partial\beta} \;=-\; \frac{n}{\hat{\beta}} \;+\; \frac{\sum_{i=1}^{n} t_i^{\hat{\alpha}}}{\hat{\beta}^2} \;=\; 0 \qquad (9)$$

And due to the high degree of non-linearity in equation (8), it cannot be solved by conventional methods; therefore, it can be solved using the Newton–Raphson method (Harter & Moor, 1965), as follows:

$$\hat{\alpha}_j = \hat{\alpha}_{j-1} - \frac{g(\hat{\alpha}_{j-1})}{g'(\hat{\alpha}_{j-1})} \qquad (10)$$

Where :

$$g(\hat{\alpha}) = \frac{\sum_{i=1}^{n} t_i^{\tilde{\alpha}}\;\ln t_i}{\sum_{i=1}^{n} t_i^{\tilde{\alpha}}} - \frac{1}{\hat{\alpha}} - \frac{\sum_{i=1}^{n} \ln t_i}{n}$$

$$g'(\hat{\alpha}) = \frac{\partial g(\hat{\alpha})}{\partial\hat{\alpha}}$$

$$= \frac{\sum_{i=1}^{n} t_i^{\hat{\alpha}} \sum_{i=1}^{n} t_i^{\hat{\alpha}} (\ln t_i)^2 - (\sum_{i=1}^{n} t_i^{\hat{\alpha}} \ln t_i)^2}{(\sum_{i=1}^{n} t_i^{\hat{\alpha}})^2} + \frac{1}{\hat{\alpha}^2}$$

If $\hat{\beta}_{mL}$, $\hat{\alpha}_{mL}$ are the maximum likelihood estimators of the parameters β , α, and if $h(\alpha,\beta)$ is one of the functions in the parameter space (Ω), then $h(\hat{\alpha}_{mL}\,,\,\hat{\beta}_{mL})$ is the maximum likelihood estimator of the function $h(\alpha,\beta)$ if the mapping is (1-1).

Since maximum likelihood estimators are characterized by the property of invariance, therefore, by using this property, we obtain an estimator of the reliability function as follows:

$$\hat{R}_{mL}(t) = exp(-\frac{t^{\hat{\alpha}_{mL}}}{\hat{\beta}_{mL}}) \qquad (11)$$

And similarly, an estimator of the hazard function can be obtained .

$$\hat{h}_{mL}(t) = \frac{\hat{\alpha}_{mL}}{\hat{\beta}_{mL}} t_i^{\hat{\alpha}_{mL}-1} \qquad (12)$$

2.2. Method of Moments (MoM) (Montgomery & Runger, 2010)

This method is characterized by its simplicity and is based on equating the population moments with the sample moments in order to estimate the distribution parameters μ_k with the sample moment (m_k). This approach leads to the derivation of estimators for the parameters, which has made it one of the most commonly used methods for parameter estimation.

The k-th moment of the Weibull distribution is:

$$\mu_k' = (\beta)^{\frac{k}{\alpha}} \Gamma(1 + \frac{k}{\alpha})$$

From this, the first and second moments can be obtained, which are:

$$\mu_1' = \beta^{\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha})$$

$$\mu_2' = \beta^{\frac{2}{\alpha}} \Gamma(1 + \frac{2}{\alpha}) \quad (13)$$

From this, the mean and variance of the Weibull distribution can be obtained, where the first moment represents the mean:

$$Mean = \mu_1'$$

And the variance is :

$$Var = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \beta^{\frac{2}{\alpha}} \Gamma(1 + \frac{2}{\alpha}) - [\beta^{\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha})]^2$$

$$= \beta^{\frac{2}{\alpha}} [\Gamma(1 + \frac{2}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})]$$

From this, the coefficient of variation (C.V) can be obtained (G., 1971).

$$C.V = \sqrt{\frac{Var}{mean^2}} = \sqrt{\frac{\mu_2}{\mu_1'^2}}$$

$$C.V = \sqrt{\frac{\Gamma\left(\frac{\alpha+2}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right)}{\Gamma^2\left(\frac{\alpha+1}{\alpha}\right)}} \quad (14)$$

From the sample data, an estimator of the coefficient of variation can be obtained as follows:

$$\hat{C}.V = \sqrt{S^2/\bar{t}^2}$$

Where:

$\bar{t}$: represents the sample mean $\bar{t} = \sum_{i=1}^{n} \frac{t_i}{n}$.

S2: represents the sample variance $S^2 = \frac{\sum_{i=1}^{n} t_i^2 - n\bar{t}^2}{n-1}$

By comparing the value of the coefficient of variation with a specific table (Sinha & Kale, 1980), an estimate of the parameter α can be obtained, where the mentioned table contains a set of values of α that are substituted into equation (14) to obtain values of the coefficient of variation (C.V). Then, the estimated value of α is used in equation (13) to obtain an estimator of the parameter β, as follows:

$$\bar{t} = \hat{\beta}^{\frac{1}{\hat{\alpha}}} \Gamma(1 + \frac{1}{\overset{\Delta}{\alpha}})$$

$$\hat{\beta}^{\frac{1}{\hat{\alpha}}} = \frac{\bar{t}}{\Gamma(1 + \frac{1}{\hat{\alpha}})}$$

$$\hat{\beta}_{mom} = [\frac{\bar{t}}{\Gamma(1 + \frac{1}{\hat{\alpha}_{mom}})}]^{\hat{\alpha}_{mom}} \quad (15)$$

2.3. Least Squares Method (LSM) (O., 2010)

This method is based on the existence of a relationship between two or more variables. Through the Weibull distribution, a relationship between variables is obtained to estimate the parameters of the distribution using this method, as follows:

Taking the natural logarithm of the reliability function, we obtain:

$$\ln (R(t_i)) = - \frac{t_i^{\alpha}}{\beta}$$

By multiplying both sides by (−1), we obtain:

$$-\ln [\mathrm{R}(t_i)] = \frac{t_i^{\alpha}}{\beta}$$

$$\ln [\mathrm{R}(t_i)]^{-1} = \frac{t_i^{\alpha}}{\beta}$$

$$\ln [\frac{1}{R(t_i)}] = \frac{t_i^{\alpha}}{\beta}$$

And after taking the natural logarithm of both sides again, we obtain:

$$ln \left[\ln \frac{1}{R(t_i)}\right] = \alpha \ln \mathrm{t}_i - \ln \beta \qquad (16)$$

The relationship (16) can be transformed into a linear regression model as follows:

$$Ey_i = \mathrm{a} + \mathrm{b}\, \mathrm{x}_i \qquad (17)$$

Where :

$$y_i = \ln [\ln \frac{1}{R(t_i)}]$$

$$a = - \ln (\beta)$$

$$b = \alpha$$

$$x_i = \ln \mathrm{t}_i$$

The values of the variable y_i can be obtained using one of the non-parametric methods for estimating the reliability function $R(t_i)$ as shown in Table (1), from which the parameters of the linear model (17) can be estimated using the least squares method, as follows:

$$\hat{a}_{\mathrm{LS}} = \bar{\mathrm{y}} - \hat{\mathrm{b}}\bar{x}$$

$$\hat{b}_{\mathrm{LS}} = \frac{\sum_{i=1}^{n} y_i x_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}} \qquad (18)$$

Also, estimators for both the shape and scale parameters can be obtained using the least squares estimators $\hat{\mathrm{a}}_{\mathrm{LS}}, \hat{b}_{LS}$ for the Weibull distribution, as follows:

$$\hat{\beta}_{LS} = \mathrm{e}^{-\hat{\mathrm{a}}_{\mathrm{LS}}}$$

$$\hat{\alpha}_{LS} = \hat{\mathrm{b}}_{\mathrm{LS}} \qquad (19)$$

The researcher proposed two methods for obtaining an approximate estimate of the reliability function using the least squares method, as follows:

First proposal: By substituting the least squares estimators of the Weibull distribution parameters into the reliability function form, as follows:

$$\hat{R}_{LS}(t_i) = \exp\left(\frac{-\mathrm{t}_i^{\hat{\alpha}_{\mathrm{LS}}}}{\hat{\beta}_{\mathrm{LS}}}\right) \qquad (20)$$

Second proposal: An estimator of the observation $\hat{y}_i$ was obtained after estimating the regression model (17), and by substituting the least squares estimators of both a and b in model (17) , that is:

$$\hat{y}_i = \hat{a} + \hat{b}\, x_i$$

And by using these estimators ($\hat{y}_i$) and substituting them into the relationship between the variable $\hat{y}_i$ and the reliability function, we obtain:

$$\hat{y}_{iLS} = ln\ [\ln \frac{1}{\hat{R}_{LS}(t_i)}]$$

And by taking the natural logarithm of both sides of the above relationship, we obtain:

$$e^{\hat{y}_i} = \ln \frac{1}{R_{LS}(t_i)}$$

And by taking the exponential (antilogarithm) of both sides of the above equation again, we obtain:

$$exp\ [\exp(\hat{y}_{iLS})] = \frac{1}{\hat{R}_{LS}(t_i)}$$

And by rearranging the above equation, estimators of the reliability function are obtained.

$$\hat{R}_{LS}(t_i) = \frac{1}{\exp[\exp(\hat{y}_{LS})]} \quad (21)$$

Table (1) non-parametric methods for estimating the reliability function (Nelson, 1982)

Method	$\hat{R}(t_i)$
Mean Rank	1- i / n + 1
Median Rank	$1 - \frac{i - 0.3}{n + 0.4}$
Symmetrical R(ti)	$1 - \frac{1 - 0.5}{n}$

Practical Aspect: Simulation Experiments and Evaluation of the Efficiency of the Estimators

The practical aspect of the research will rely on designing an intensive simulation experiment to evaluate the estimation performance of the three methods presented in the theoretical aspect, namely (Maximum Likelihood, Method of Moments, and Least Squares methods), through estimating the two parameters of the Weibull distribution (the shape parameter and the scale parameter). Stata 17 will be used to implement the simulation according to the Mata language in order to write efficient and fast simulation codes, as follows:

A comparative study of the estimators of the Weibull distribution using simulation in STATA, as follows:

- The shape parameter, denoted by β in the theoretical aspect, takes four values, and each value represents one of the failure patterns (Meeker & Escobar, 1998) according to Table 2.

Table (2) The relationship between the values of the shape parameter β and the failure patterns in the Weibull distribution

Failure patterns	**Dominant pattern**	**valueβ**
Early failure (decreasing hazard rate)	$\beta < 1$	**0.5**
Random failure (constant hazard rate)	$\beta = 1$	**1.0**
Moderate Wear-out failure (increasing hazard rate)	$\beta > 1$	**2.0**
Accelerated Wear-out failure (rapidly increasing hazard rate)	$\beta > 1$	**3.0**

- The scale parameter, denoted as α, was set at 1000 to avoid small numerical values and to expand the range, ensuring the simulation is realistic and yields logical failure times-such as hours rather than minutes, seconds, or fractions of a second. Fundamentally, it does not affect the distribution type but rather influences the units of measurement, as established in reliability literature (Abernethy, 2006).

The sample size (n) assumed four distinct values to graduate from small to large samples, aiming to accurately observe the behavior of the estimated parameters without the influence of asymptotic properties (Mooney, 1997), as follows: (small sample = 20, large sample = 200, and intermediate samples of 50 and 100, respectively).

- The number of replications (R) was fixed at 1000 for each scenario to achieve a standard deviation of less than 5% when estimating the Mean Square Error (MSE) (Meeker & Escobar, 1998). According to the following code:

```
Clear all
Set more off
//1. DATA GENERATION FUNCTION (Inverse Transform Method)
Function Weibull_rnd (real scalar beta, real scalar eta, real scalar n)
{U = runiform (n,1)
t = eta : * (- log (1: -U)) : ^ (1:/ beta)
return (t)}
```

The estimation functions will be programmed within the simulation loop; the Maximum Likelihood Estimator (MLE) will be formulated using the simplified Newton-Raphson method with a numerical solution, while the Method of Moments Estimator (MME) will be obtained by solving the coefficient of variation equation using the Bisection algorithm. As for the Least Squares Estimator (LSE), it will be based on median ranks; thus, the value of R(t) at each observed failure time will be estimated via a non-parametric method and subsequently transformed into a linear form to apply regression analysis (O., 2010), according to the following formula:

$$R(t) = e^{-\left(\frac{t}{\alpha}\right)^{\beta}} \tag{22}$$

```
// 2. ESTIMATION METHODS
//2.1 Maximum Liklihood Estimation (MLE) using Newton- Raphson function mel-weibull (t) {
 n= rows
logt = log (t)
beta_hat = 1.0 // Initial starting value
for (iter =1; iter <=20; iter++) {
sum_t_beta = sum(t:^beta_hat)
sum_t_beta_log = sum((t:^beta_hat) :* logt)
score = n/beta_hat + sum(logt) – n/sum_t_beta * (sum_t_beta * sum((t:^beta_hat) : *(log:^2)) – sum_t_beta_logt^2)
beta_new = beta_hat – score / hees
if (beta_new <= 0.1) beta_new = 0.1
if (abs(beta_new – beta_hat) < 1e-6) break
beta_hat = beta_new {
eta_hat = (sum(t :^beta_hat) / n)^(1/beta_hat)
return(beta_hat , eta_hat)
//2.2 Method of Moments (MoM) using Bisection Method for Gamma function
Function mle_weibull(t) {
n= rows(t)
tbar = mean(t)
cv = sqrt(variance(t)) / tbar
// Internal function to find root of CV
Lo = 0.1 ; hi =10.0
for(i=1; i<=50; i++) {
mid = (lo + hi) / 2
cv_mid = sqrt (gamma(1 + 2/mid) / (gamma(1 + 1/mid)^2) – 1)
if cv_mid > cv_sample) lo = mid
else hi = mid      {
beta_hat = (lo + hi) / 2
eta_hat = tbar / gamma(1 + 1/beta_hat)
return(beta_hat , eta_hat) {
// 2.3 Least Squares Method (LSM) using Median 1sm_weibull(t) {
n= rows(t)
scorted_t = sort(t , 1)
i= (1: :n)
F=hat = (I : - 0.3) :/ (n :+0.4) //Bernard's Approximation
y= log(-log(1 : - F_hat))
x= log(scorted_t)
X = (J(n, 1, 1) , x)
b= invsym(cross(X,X) * cross(X,y)
beta_hat = b[2]
eta_hat = exp(-b[1] / b[2])
return (beta_hat, eta_hat) {
```

Calculating of simulation parameters

```
true_betas =(0.5, 1.0, 2.0, 3,0)
true_eta = 1000.0
sampie_sizes = (20, 50, 100, 200)
N_reps = 1000
results = J(0, 9, .)
printf("{txt}starting simulation...\n")
for (b=1; b<=1ength(true_betas); b++) {
beta0 = true_betas[b]
for (s=1; s<=1ength(sample_sizes); s++
n0 = sample_sizes[s]
m_mel = m_mom = m_1sm
for (r=1; r<=reps, r++)  {
data = Weibull_rnd(beta0, true_eta, n0)
m_m1e[r, .] = m1e_weibull(data)
//columns: beta_true, n, method, mean_beta, bias_beta, mse_beta, mean_eta, bias_eta, mse_eta
all results = J(0, 9, .)
for (b_idx=1; b_idx<=1ength(true_betas); b_idx++ {
beta0 = true_betas[b_idx]
for (n_idx=1; n_idx<=1ength(sample_sizes); n_idex++) {
n0 = sample_sizes[n_idx]
```

Aggregating replication results for each group according to the following code:

```
res_MLE = J(N_reps, 2, .)
 res_MOM = J(N_reps, 2, .)
res_LSM = J(N_reps, 2, .)
{bh, eh} = mle_weibull(t)
res_MLE[rep, .] = (bh, eh)
{bh, eh} = mom_weibull(t)
res_MOM[rep, .] = (bh, eh)
{bh, eh} = lsm_weibull(t)
 res_LSM[rep, .] = (bh, eh)  {
```

Calculation of evaluation criteria for each method and displaying them in the results according to the following code:

```
function calc_metrics(estimates, true_val) {
mean_est = mean(estimates)
bias = mean_est – true_val
mse = mean((estimates :- true_val):^2)
return(mean_est, bias, mse) {
//MLE
{mb, biasb, mseb} = calc_metrics(res_MLE[,1], beta)
{mb, biasb, mseb} = calc_metrics(res_MLE[,2], true_eta)
all_results = all_results \ (beta0, n0, 1, biasb, mseb, me, biase, msee)
//MoM
{mb, biasb, mseb} = calc_metrics(res_MoM[,1], beta)
{mb, biasb, mseb} = calc_metrics(res_MoM[,2], true_eta)
all_results = all_results \ (beta0, n0, 2, biasb, mseb, me, biase, msee)
//LSM
{mb, biasb, mseb} = calc_metrics(res_LSM[,1], beta)
{mb, biasb, mseb} = calc_metrics(res_LSM[,2], true_eta)
all_results = all_results \ (beta0, n0, 3, biasb, mseb, me, biase, msee) {
```

3. Result and Discusion

3.1. Presentation of Results

After extracting the Stata 17 results according to the codes described in the previous sections, the following results were obtained :

Table (3): Mean Square Error (MSE) for the Estimation of Parameter β

LSM (β)	MoM (β)	MLEβ)	n	β
0.0398	0.0587	0.0421	20	0.5
0.0161	0.0205	0.0153	50	
0.0083	0.009	0.0072	100	
0.0042	0.0049	0.0035	200	

0.0495	0.0764	0.0513	20	1.0
0.0199	0.0281	0.0196	50	
0.0102	0.0139	0.0097	100	
0.0052	0.0069	0.0048	200	
0.2207	0.3312	0.2124	20	2.0
0.853	0.1194	0.0826	50	
0.0429	0.0572	0.0411	100	
0.0216	0.0284	0.205	200	
0.5348	0.8235	0.5121	20	3.0
0.2088	0.2857	0.1962	50	
0.0976	0.1301	0.0934	100	
0.0480	0.0627	0.0458	200	

Source: Prepared by the researcher based on Stata 17 software results

Table (4): Bias for the Estimation of the Shape Parameter β

LSM (β)	MoM (β)	MLEβ)	n	β
0.0228	0.0315	0.0182	20	0.5
0.0087	0.0124	0.0068	50	
0.0038	0.0057	0.0029	100	
0.0017	0.0026	0.0011	200	
-0.0135	-0.0428	-0.0097	20	1.0
-0.0052	-0.0171	-0.0032	50	
-0.0023	-0.0083	-0.0015	100	
-0.0010	-0.0041	-0.0007	200	
0.0521	0.0987	0.0356	20	2.0
0.0194	0.0395	0.0128	50	
0.0082	0.0178	0.0054	100	
0.0040	0.0083	0.0021	200	
0.0812	0.1604	0.0627	20	3.0
0.0310	0.0593	0.0215	50	
0.0135	0.0256	0.0089	100	
0.0063	0.0119	0.0041	200	

Source: Prepared by the researcher based on Stata 17 software results

Table (5): Mean Square Error (MSE) for the Estimation of the Scale Parameter (True α = 1000)

LSM (α)	MoM (α)	MLEα)	n	β
19221.4	21387.6	18234.1	20	0.5
7044.1	7526.3	6431.2	50	
3210.9	3381.1	2987.6	100	
1497.2	1562.8	1389.3	200	
21336.7	24872.5	201444.3	20	1.0
9441.3	9245.6	7932.8	50	
3912.6	4190.0	3708.5	100	
1815.8	1910.7	1723.2	200	
9742.8	13122.4	9267.5	20	2.0
3516.1	4527.3	3312.9	50	
1648.7	2001.8	1562.1	100	
772.0	919.5	731.4	200	
6833	10115.2	6477.9	20	3.0
2421.8	3387.6	2271.5	50	
1128.6	1487.2	1074.3	100	
527.3	675.4	506.2	200	

Source: Prepared by the researcher based on Stata 17 software results

3.2. Analysis and Interpretation of Results

Based on the results presented in the previous section, the Mean Square Error (MSE) and Bias served as the primary criteria for evaluating the performance efficiency of the three methods used to estimate the parameters of the Weibull distribution. It is noteworthy that the MSE incorporates both the squared bias and the variance, rendering it a more comprehensive metric for judging the efficiency of estimators, as follows:

First: Analysis of mean square error (MSE) For (β)

It is observed from Table 3 that the mean squared error (MSE) values decrease as the sample size increases for all methods used in the simulation. This is the consistency property of estimators that is satisfied by estimators as the sample size increases until the random error decreases and approaches the true value of the parameters (ESCWA, 2002).

The Maximum Likelihood Estimator (MLE) reduced its MSE from 0.00421 at n=20 to 0.0035 at n=200, which is nearly 91.7% less. MoM resulted in 91.6% improvement and LSM improved by 89.4% which shows that MoM is the least efficient among the three methods and (MLE) is the most efficient.

In all the scenarios studied the Method of Moments estimator exhibited the highest MSE values. At β=3.0 and n=20, the MSE of MoM was 0.8235, compared to 0.5348 for the MLE. This means that the efficiency of the Method of Moments never exceed that of the Maximum Likelihood Estimator of 62%.

This behaviour is the primary cause of the Method of Moments being unable to accurately measure the shape efficiency under high skewness (e.g. β=0.5) particularly when the distribution deviates significantly from being symmetrical. This results in a significant decrease in the accuracy of the estimation (Hogg et al., 2020).

Second: Analysis Bias for β

As seen in Table 4, bias values of Maximum Likelihood Estimator (MLE) is the smallest of the methods considered and it tends to zero as the sample size increases. The bias is very small 0.0011 for n=200. This is considered to be due to the nonlinear nature of the equations of the MLE estimation (O., 2010).

The Method of Moments estimator, on the other hand, has a large bias that depends on β. The bias is negative when β<1 and positive when β >1, indicating a systematic distortion in estimating the shape parameter. The method tends to underestimate β when there is high positive skewness, and the opposite when there is high negative skewness. This is due to the fact that the moment-based equations link the shape parameter to the coefficient of variation through a nonlinear relationship, causing the variability in the sample mean or variance to propagate and amplify in the estimation of β. This explains the poor performance of the Method of Moments, particularly when β≤1 (Bain & Antle, 1967). Finally, the Ordinary Least Squares estimator shows a bias larger than that of the MLE and smaller than that of the Method of Moments, especially as the sample size increases, for the same reasons discussed in the first point.

Third: Evaluating the Performance Efficiency of the Scale Parameter (α) Estimators

In Table (5), it is observed that the efficiency patterns of the estimators α are consistent with those of β, allowing the results to be interpreted in the same manner. The results indicate that the Maximum Likelihood Method was the best in terms of having the lowest mean squared error, followed by the Least Squares Method, while the Method of Moments ranked last as the least efficient estimator. The relatively high absolute values of the mean squared errors are attributed to the large sample size assumed in the simulation, which was set to 1000. It is also worth noting that the efficiency difference between the Maximum Likelihood Method and the Least Squares Method was small, indicating that the Least Squares Method is a good and reliable alternative to the Maximum Likelihood Method when the latter is difficult or not available to implement.

4. Conclusion

1. The Maximum Likelihood Method is considered more efficient in estimation than the Method of Moments and the Least Squares Method.
2. Transforming the reliability function into a linear form, together with the use of median ranks in estimating the survival function, results in an estimator that is relatively stable even in cases of limited data availability.
3. The bias values indicate that the Maximum Likelihood Estimator is the least biased, and its bias approaches zero as the sample size increases.
4. The Method of Moments relies solely on the first and second moments, which are statistical measures that do not efficiently capture the distributional shape when skewness is high, as in the case of β = 0.5, or when the distribution is far from symmetry, leading to a substantial loss in the estimator's accuracy.
5. The efficiency difference between the Maximum Likelihood Method and the Least Squares Method was found to be small, indicating that the Least Squares Method is a good and reliable alternative to the Maximum Likelihood Method when the latter is difficult or unavailable to apply.

6. Sample size is considered a crucial factor in selecting the estimation method. When the sample size is small, the Least Squares Method may be used without a substantial loss in estimation efficiency compared with the Maximum Likelihood Method. However, for large sample sizes, the Maximum Likelihood Method is regarded as the most efficient approach overall.

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