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NUMERICAL TREATMENT OF INTEGRAL EQUATIONS USING CHEBYSHEV COLLOCATION METHODS

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Abstract
Numerical evaluation of integral equations has become one of the important problems in numerical analysis and computational mathematics, and the calculation of approximate solutions is reduced to solving some systems of linear algebraic equations. In this paper, we will discuss the approximate solutions of linear single integral equations with a Cauchy kernel on a finite interval and Fredholm linear integral equations of the second kind using Chebyshev polynomials. Some numerical examples were presented to illustrate the method.

1. Introduction

In numerous branches of mathematics, including number theory, differential equations, approximation theory, numerical analysis, geometry, and statistics, Chebyshev polynomials play a crucial role. The definitions and fundamental characteristics of the initial, subsequent, third, and fourth kinds of Polynomials Chebyshev are shown in this section. These will aid in the construction of our key findings (Atkinson, 1997; Backer & Miller, 1977; Banas & Rzepka, 2004; Darania & Ivaz, 2008; Fox & Parker, 1968; James & Peter, 1992; Kulikov & Makarov, 2023; Mason & Handscomb, 2003; Talaei et al., 2022; and Wazwaz, 2011). A degree *n* polynomial in *x*, the Polynomials Chebyshev $T_n(a)$. The connection that follows defines the first class:

$$T_n(a) = \cos n\theta\,,\ a = \cos\theta \qquad (1.1)$$

The range of the related variable θ can be taken as [0,π] if the variable *x* range is the interval [-1,1]. Since $a = -1$ corresponds to $\theta = \pi$ and $a = 1$ corresponds to $\theta = 0$, these ranges are traversed in different ways. De Moivre's Theorem has made it widely known that cos $n\theta$ is a degree *n* polynomial. Actually, we are familiar with the following fundamental formulae:

$$\cos 0\theta = 1,\ \cos 1\theta = \cos\theta\,,\ \cos 2\theta = 2\cos^2\theta - 1$$
$$\cos\ 3\theta = 4\cos^3\ \theta - 3\cos\ \theta, \cos\ 4\theta = 8\cos^4\ \theta - 8\cos^2\ \theta + 1,\ \ldots.$$

From (1.1), we can quickly conclude that the initial several Chebyshev polynomials:

$$T_0(a) = 1, T_1(a) = x, T_2(a) = 2a^2 - 1, T_3(a) = 4a^3 - 3a, T_4(a) = 8a^4 - 8a^2 + 1, \ldots. \qquad (1.2)$$

Combining the identity of trigonometry

$$\cos\ n\theta + \cos\ (n-2)\theta = 2\cos\ \theta\cos\ (n-1)\theta$$

The basic recurrence connection is thus obtained:

$$T_n(a) = 2xT_{n-1}(a) - T_{n-2}(a),\ n = 2,3, \ldots \qquad (1.3)$$

which, when paired with the original facts

$$T_0(a) = 1,\ T_1(a) = a \qquad (1.4)$$

iteratively produces every polynomial.$\{T_n(a)\}$ very efficiently.

The Chebyshev polynomial $U_n(a)$ of The second kind is described by the following relation and is an nth degree polynomial in (a).

$$U_n(a) = \frac{\sin(n+1)\,\theta}{\sin\theta},\ a = \cos\theta \tag{1.5}$$

The ranges of x and θ are the same as for $T_n(x)$. The elementary formula gives

$$\sin\ 1\theta = \sin\ \theta,\ \sin\ 2\theta = 2\sin\ \theta\cos\ \theta,\ \sin\ 3\theta = \sin\ \theta(4\cos^2\ \theta - 1)$$
$$\sin\ 4\theta = \sin\ \theta(8\cos^3\ \theta - 4\cos\ \theta), \ldots$$

such that we can observe that the sine function ratio (1.5) is, in fact, a polynomial in cos θ, Therefore we may quickly conclude that

$$U_0(a) = 1,\ U_1(a) = 2a,\ U_2(a) = 4a^2 - 1,\ U_3(a) = 8a^3 - 4a, \tag{1.6}$$

By combining the trigonometric identity

$$\sin\ (n+1)\theta + \sin\ (n-1)\theta = 2\cos\ \theta\sin\ n\theta$$

As we discover, $U_n(a)$ satisfies the recurrence relation

$$U_n(a) = 2xU_{n-1}(a) - U_{n-2}(a),\ n = 2,3, \ldots \tag{1.7}$$

which, in addition to the basic conditions

$$U_0(a) = 1,\ U_1(a) = 2a \tag{1.8}$$

gives a method for producing the polynomials that is effective. A trigonometric identity that is comparable

$$\sin\ (n+1)\theta - \sin\ (n-1)\theta = 2\sin\ \theta\cos\ n\theta$$

brings us into a relationship.

$$U_n(a) - U_{n-2} = 2T_n(a),\ n = 2,3, \ldots \tag{1.9}$$

between first- and second-kind polynomials.

Two more families of polynomials, V_n and W_n, that are connected to T_n and U_n can be created. Their ranges for (a) and θ are the same as for $T_n(a)$, and their trigonometric definitions involve the half angle θ/2 (where $x = \cos\ \theta$ as before). The forms three and four of Polynomial Chebyshev, $V_n(a)$ and $W_n(a)$,, are polynomials of degree n in a that are, respectively, defined by

$$V_n(a) = \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta} \tag{1.10}$$

and

$$W_n(a) = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta} \tag{1.11}$$

where $a = \cos\ \theta$.

We may easily demonstrate that

$V_0(a) = 1,\ V_1(a) = 2a - 1,\ V_2(a) = 4a^2 - 2a - 1,\ V_3(a) = 8a^3 - 4a^2 - 4a + 1, \ldots,$ (1.12)

and

$W_0(a) = 1,\ W_1(a) = 2a + 1,\ W_2(a) = 4a^2 + 2a - 1,\ W_3(a) = 8a^3 + 4a^2 - 4a - 1, \ldots.$ (1.13)

A recurrence relation may also be used to effectively produce these polynomials since

$$\cos\left(n + \frac{1}{2}\right)\theta + \cos\left(n - 2 + \frac{1}{2}\right)\theta = 2\cos\ \theta\cos\left(n - 1 + \frac{1}{2}\right)\theta$$

and

$$\sin\left(n + \frac{1}{2}\right)\theta + \sin\left(n - 2 + \frac{1}{2}\right)\theta = 2\cos\ \theta\sin\left(n - 1 + \frac{1}{2}\right)\theta$$

It follows instantly that

$$V_n(a) = 2aV_{n-1}(a) - V_{n-2}(a),\ n = 2,3, \ldots \tag{1.14}$$

and

$$W_n(a) = 2aW_{n-1}(a) - W_{n-2}(a),\ n = 2,3, \ldots \tag{1.15}$$

with

$$V_0(a) = 1,\ V_1(a) = 2a - 1 \tag{1.16}$$

and

$$W_0(a) = 1,\ W_1(a) = 2a + 1 \tag{1.17}$$

Thus $V_n(a)$ and $W_n(a)$ have the exact same recurrence relationship as$T_n(a)$ and $U_n(a)$, and the only difference between their generations is how the starting condition is prescribed for n=1. For further details, see (Wazwaz, 2011; Ghafil, Al-Juaifri, & Al-Haboobi, 2024; Al-Juaifri, Ghafil, & Al-Haboobi, n.d; Khraibet & Ghafil, 2021; Sharhan & Al-Muslimawi, 2023; and Al-Bahrani & Al-Muslimawi, 2025).

2. Method

2.1 The general range of Chebyshev polynomials [t,r]

More broadly, by aligning this range with the range [-1,1] of the linear transformation of a new variable s, For each given finite range, we may define Chebyshev polynomials. [t,r] of a.

$$s = \frac{2x - (t + r)}{r - t} \tag{2.1}$$

Thus, $T_n(s)$, where s is given by (1.2), is the first kind Chebyshev polynomial appropriate for $[t, r]$. In a similar manner, and $U_n(s), V_n(s)$ and $W_n(s)$ are the second, third, and fourth kind polynomials acceptable for $[t, r]$.

2.2 The general range of Chebyshev polynomials [t,r]

This particular area examines the approximation of the Cauchy kernel SIE solution using the polynomial Chebyshev, and provides a numerical example to demonstrate the approach (James & Peter, 1992; and Mason & Handscomb, 2003). Consider the following SIE:

$$\int_{-1}^{1} \frac{K(a,c)}{c-a}\varphi(c)dc + \int_{-1}^{1} L(a,c)\varphi(c)dc = f(a), \ -1 < a < 1 \quad (2.2.1)$$

where $\varphi(a)$ is an unknown function that needs to be found and real-valued continuous functions $K(a,c), L(a,c)$ and $f(a)$ are given.

The characteristic singular integral equation, often known as the most basic integral equation of type (2.2.1), is expressed in the following form:

$$\int_{-1}^{1} \frac{\varphi(c)}{c-a} dc = f(a), \ -1 < a < 1 \quad (2.2.2)$$

and $L(a,c) = 0. K(a,c) = 1$

The full analytical answers to the equation (2.2.2) are given by the following cases:

Case (I): The solution $\varphi(a)$ is unbounded at both the endpoints $a = \pm 1$

$$\varphi(a) = -\frac{1}{\pi^2\sqrt{1-a^2}} \int_{-1}^{1} \frac{\sqrt{1-c^2}f(c)}{c-a} dc + \frac{A}{\sqrt{1-a^2}} \quad (2.2.3)$$

where A is an arbitrary constant.

Case (II): The solution $\varphi(a)$ is bounded at both the endpoints $a = \pm 1$

$$\varphi(a) = -\frac{\sqrt{1-a^2}}{\pi^2} \int_{-1}^{1} \frac{f(c)}{\sqrt{1-c^2}(c-a)} dc \quad (2.2.4)$$

The following condition must be met for the answer to exist:

$$\int_{-1}^{1} \frac{f(c)}{\sqrt{1-c^2}} dc = 0 \quad (2.2.5)$$

Case (III): The solution $\varphi(a)$ is unbounded at the endpoint $a = -1$ but bounded at the endpoint $a = +1$

$$\varphi(a) = -\frac{1}{\pi^2}\sqrt{\frac{1-a}{1+a}} \int_{-1}^{1} \sqrt{\frac{1+c}{1-c}} \frac{f(c)}{c-a} dc \quad (2.2.6)$$

Case (IV): The solution $\varphi(a)$ is limited toward the conclusion $a = -1$ but unbounded at the endpoint $a = +1$

$$\varphi(a) = -\frac{1}{\pi^2}\sqrt{\frac{1+a}{1-a}} \int_{-1}^{1} \sqrt{\frac{1-c}{1+c}} \frac{f(c)}{c-a} dc \quad (2.2.7)$$

The unknown function $\varphi(c)$ in equation (2.2.1) can be stated in the way outlined below:

$$\varphi(a) = \frac{\Phi_k(a)\Psi_k(a)}{\sqrt{1-a^2}}, \ (k = 1,2,3,4) \quad (2.2.8)$$

where $\Phi_k(a)$ is a function of a that behaves nicely within the interval $a \in [-1,1]$, and $\Psi_1(a) = 1$ in case $(I), \Psi_2(a) = 1 - a^2$ in case $(II), \Psi_3(a) = 1 - a$ in case (III) and $\Psi_4(a) = 1 + a$ in case (IV). Let the unidentified function $\Phi_k(a)$ be roughly expressed as follows using a polynomial of degree n:

$$\Phi_k(a) \approx \sum_{j=0}^{n} M_j^{(k)} a^j, \ (k = 1,2,3,4) \quad (2.2.9)$$

where $M_j^{(k)}$ are arbitrary values.

Substituting from (2.2.8) and (2.2.9) into equation (2.2.1) we obtain:

$$\sum_{j=0}^{n} M_j^{(k)} \left[\int_{-1}^{1} \frac{\Psi_k(c)P(a,c)c^j}{\sqrt{1-c^2}(c-a)} dc + \int_{-1}^{1} \frac{\Psi_k(c)L(a,c)c^j}{\sqrt{1-c^2}} dc\right] = f(a), (k = 1,2,3,4), -1 < a < 1. \quad (2.2.10)$$

By using the following "Chebyshev approximations" to the degenerate kernels $P(a,c)$ and $L(a,c)$

$$\left.\begin{aligned} P(a,c) &\approx \sum_{p=0}^{m} P_p(a)t^p \\ L(a,c) &\approx \sum_{q=0}^{s} L_q(a)t^q \end{aligned}\right\} \quad (2.2.11)$$

with known functions $P_p(a)$ and $L_q(a)$, we obtain the following relation with the unknown constants $M_j^{(k)}, (j = 0,1,2,\dots,n)$

$$\sum_{j=0}^{n} M_j^{(k)} \left[\sum_{p=0}^{m} P_p(a) \int_{-1}^{1} \frac{\Psi_k(c)c^{p+j}}{\sqrt{1-c^2}(c-a)} dc + \sum_{q=0}^{s} L_q(a) \int_{-1}^{1} \frac{\Psi_k(c)c^{q+j}}{\sqrt{1-c^2}} dc\right] = f(a) \quad (2.2.12)$$

One way to express equation (2.2.12) is as follows:

$$\sum_{j=0}^{n} M_j^{(k)} \left[\sum_{p=0}^{m} P_p(a)\Omega_{p+j}^{(k)}(a) + \sum_{q=0}^{s} L_q(a)\gamma_{q+j}^{(k)}\right] = f(a),\ (k = 1,2,3,4),\ -1 < a < 1 \quad (2.2.13)$$

where

$$\Omega_{p+j}^{(k)}(a) = \int_{-1}^{1} \frac{\Psi_r(c)c^{p+j}}{\sqrt{1-c^2}(c-a)} dc \quad (2.2.14)$$

and

$$\gamma_{q+j}^{(k)} = \int_{-1}^{1} \frac{\Psi_k(c)c^{q+j}}{\sqrt{1-c^2}} dc \quad (2.2.15)$$

Making use of zerosa_P of the polynomial Chebyshev $T_{n+1}(a) = \cos((n+1)\cos^{-1}(a))$ into a formula (2.2.13) the system we receive is a set of linear algebraic formulas that contain unknown constants $M_j^{(k)}$:

$$\sum_{j=0}^{n} M_j^{(k)}\alpha_j(a_P) = f(a_P),\ (k = 1,2,3,4) \quad (2.2.16)$$

where

$$a_P = \cos\left(\frac{(2P-1)}{2(n+1)}\pi\right),\ P = 1,2,3,\dots,n+1 \quad (2.2.17)$$

and

$$\alpha_j(a_P) = \sum_{p=0}^{m} K_p(a_P)\Omega_{p+j}^{(k)}(a_P) + \sum_{q=0}^{s} L_q(a_P)\gamma_{q+j}^{(k)} \quad (2.2.18)$$

By resolving the linear algebraic equation system (3.16) the unknown constants are obtained. $c_j^{(r)}$ and substituting into the relations (2.2.8) and (2.2.9). We arrive at the approximate answer to equation (2.2.1) as follows:

$$\varphi(a) \approx \frac{\Psi_k(a)}{\sqrt{1-a^2}} \sum_{j=0}^{n} M_j^{(k)} a^j, \ (k = 1,2,3,4) \tag{2.2.19}$$

2.3 Approximate solution of linear Fredholm integral equations

This section covers the numerical solution of second-kind Fredholm integral equations using Chebyshev polynomials of the first, second, third, and fourth kinds T_n (a), U_n (a)and W_n (a). The Fredholm integral equation is converted into a matrix equation using this technique. Examine the following equation for the Fredholm integral:

Take a look at this Fredholm integral equation:

$$\phi(a) = g(a) + \int_a^b \mathrm{P}(a,c)\phi(c)dc, \ a \le a \le b \tag{2.3.1}$$

where $g(a), P(a,c)$ are given functions, a and c are actual variables that change throughout the timeframe$[t,r]$ and $\phi(a)$ is the function that has to be ascertained.

Let the unknown function $\phi(a)$ in equation (2.3.1) be approximated by the polynomial function $\phi_n(a)$:

$$\phi_n(a) = \sum_{i=0}^{n} \alpha_i^{(m)} G_i^{(m)}(a), \ (m = 1,2,3,4) \tag{2.3.2}$$

where $\alpha_i^{(m)}, i = 0,1,\dots,n$ are the coefficients that are unknown and $G_i^{(1)}(a) = T_i(a)$, $G_i^{(2)}(a) = U_i(a)$, $G_i^{(3)}(a) = V_i(a)$ and $G_i^{(4)}(a) = W_i(a)$. Substituting from (2.3.2) into (2.3.1) we obtain

$$\sum_{i=0}^{n} \alpha_i^{(m)} G_i^{(m)}(a) = g(a) + \int_a^b \mathrm{P}(a,c) \sum_{i=0}^{n} \alpha_i^{(m)} G_i^{(m)}(c)dc, \ (m = 1,2,3,4) \tag{2.3.3}$$

Hence the residual equation is defined as:

$$R_n(a) = \sum_{i=0}^{n} \alpha_i^{(m)} G_i^{(m)}(a) - \int_a^b \mathrm{P}(a,c) \sum_{i=0}^{n} \alpha_i^{(m)} G_i^{(m)}(c)dc - g(a) \tag{2.3.4}$$

The unknown coefficients $\alpha_i^{(m)}$ are defined by selected several collocation points a_j so that $R_n(a_j) = 0$ for j from 0 to n. The collocation points are chosen from the interval $[t,t]$ as follows:

$$a_j = u + \frac{j(r-t)}{n}, \ j = 0,1,\dots,n \tag{2.3.5}$$

Thus, it is possible to transform integral equation (4.3) into a matrix equation $T_n X = R_n$ where

$$T_n = \left[G_i^{(m)}(a_j) - \int_t^r P(a_j,c) G_i^{(m)}(c)dc \right]_{j=0}^{n}, \ i = 0,1,\dots,n; m = 1,2,3,4$$

$$X^T = \left[\alpha_i^{(m)}\right]_{i=0}^{n}, \ (m = 1,2,3,4). \tag{2.3.6}$$

3. Result and Discussion

3.1. Application on Cauchy Kernel Equations

The integral equation that follows should be considered

$$\int_{-1}^{1}\left[\frac{1}{c-a}+ac\right]\varphi(c)dc=a+a^3,\ -1<a<1 \tag{3.1.1}$$

which is the same as equation (2.2.1) in case of $P(a,c)=1, L(a,c)=ac$ and $f(a)=a+a^3$.

One way to express equation (3.1.1) is as follows:

$$\int_{-1}^{1}\frac{\varphi(c)}{c-a}dc=(1+\lambda)a+a^3 \tag{3.1.2}$$

where

$$\lambda=-\int_{-1}^{1}t\varphi(t)dt \tag{3.1.3}$$

λ is a constant should be determined in each case.

From (3.1.1), (3.1.2), (2.2.5), (2.2.6),(2.2.7) and (2.2.8) For the four scenarios listed above, we get the precise solutions of equation (2.2.1) as follows:

$$\text{Case (I)}: \varphi(a)=\frac{A}{\sqrt{1-a^2}}+\frac{1}{\pi\sqrt{1-a^2}}\left[a^4+\frac{a^2}{2}-\frac{5}{8}\right] \tag{3.1.4}$$

where the constant A is arbitrary.

$$\text{Case (II)}: \varphi(a)=-\frac{1}{\pi}\sqrt{1-a^2}\left[a^2+\frac{3}{2}\right]. \tag{3.1.5}$$

$$\text{Case (III)}: \varphi(a)=-\frac{1}{\pi}\sqrt{\frac{1-a}{1+a}}\left[a^3+a^2+\frac{3}{2}a+\frac{3}{2}\right]. \tag{3.1.6}$$

$$\text{Case (IV)}: \varphi(a)=\frac{1}{\pi}\sqrt{\frac{1+a}{1-a}}\left[a^3-a^2+\frac{3}{2}a-\frac{3}{2}\right]. \tag{3.1.7}$$

We now compute equation (3.1.1) approximate solution.

Since $P(a,c)=1$ and $L(a,c)=ac$, so that we have

$$\left.\begin{matrix}P_0(a)=1, & P_1(a)=P_2(a)=\cdots=P_m(a)=0\\ L_1(a)=a, & L_0(a)=L_2(a)=\cdots=L_s(a)=0\end{matrix}\right\} \tag{3.1.8}$$

Substituting from (3.1.6) into (2.2.16) we obtain:

$$\alpha_j(a_P)=\Omega_j^{(k)}(a_P)+a_P\gamma_{j+1}^{(k)},\ (j=0,1,2,\dots,n) \tag{3.1.9}$$

Changing (3.1.8) to (2.2.18) The following set of linear algebraic equations is the result:

$$\sum_{j=0}^{n}M_j^{(k)}\left[\Omega_j^{(k)}(a_P)+a_P\gamma_{j+1}^{(k)}\right]=a_P+a_P^3 \tag{3.1.10}$$

Where
$a_P=\cos\left(\frac{(2P-1)}{2(n+1)}\pi\right),\ (k=1,2,3,4),\ P=1,2,3,\dots,n+1$

It is easy to calculate $\Omega_j^{(k)}(a)$ and $\gamma_{j+1}^{k)}$ from(2.2.14) and (2.2.15) as follows:

Table 1. Values of the Coefficients Obtained from the Chebyshev Approximation

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$\gamma_j^{(1)}$	π	0	$\frac{\pi}{2}$	0	$\frac{3\pi}{8}$	0	$\frac{5\pi}{16}$
$\gamma_j^{(2)}$	$\frac{\pi}{2}$	0	$\frac{\pi}{8}$	0	$\frac{\pi}{16}$	0	$\frac{5\pi}{128}$
$\gamma_j^{(3)}$	π	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$\frac{3\pi}{8}$	$-\frac{5\pi}{16}$	$\frac{5\pi}{16}$
$\gamma_j^{(4)}$	π	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3\pi}{8}$	$\frac{3\pi}{8}$	$\frac{5\pi}{16}$	$\frac{5\pi}{16}$

Table 2. Computed Parameters for the Numerical Solution of the Singular Integral Equation

	$\Omega_j^{(1)}$	$\Omega_j^{(2)}$	$\Omega_j^{(3)}$
$j=0$	0	$-\pi a$	$-\pi$
$j=1$	π	$-\pi\left(a^2-\frac{1}{2}\right)$	$-\pi(a-1)$
$j=2$	πa	$-\pi\left(a^3-\frac{a}{2}\right)$	$-\pi\left(a^2-a+\frac{1}{2}\right)$
$j=3$	$\pi\left(a^2+\frac{1}{2}\right)$	$-\pi\left(a^4-\frac{a^2}{2}-\frac{1}{8}\right)$	$-\pi\left(a^3-a^2+\frac{a}{2}-\frac{1}{2}\right)$
$j=4$	$\pi\left(a^3+\frac{a}{2}\right)$	$-\pi\left(a^5-\frac{a^3}{2}-\frac{a}{8}\right)$	$-\pi\left(a^4-a^3+\frac{a^2}{2}-\frac{a}{2}+\frac{3}{8}\right)$
$j=5$	$\pi\left(a^4+\frac{a^2}{2}+\frac{3}{8}\right)$	$-\pi\left(a^6-\frac{a^4}{2}-\frac{a^2}{8}-\frac{1}{16}\right)$	$-\pi\left(a^5-a^4+\frac{a^3}{2}-\frac{a^2}{2}+\frac{3a}{8}-\frac{3}{8}\right)$

Table 3. Numerical Results of the Approximate Solution Using Chebyshev Polynomials

j	$\Omega_j^{(4)}$
$j=0$	π
$j=1$	$\pi(a+1)$
$j=2$	$\pi\left(a^2+a+\frac{1}{2}\right)$
$j=3$	$\pi\left(a^3+a^2+\frac{a}{2}+\frac{1}{2}\right)$
$j=4$	$\pi\left(a^4+a^3+\frac{a^2}{2}+\frac{a}{2}+\frac{3}{8}\right)$
$j=5$	$\pi\left(a^5+a^4+\frac{a^3}{2}+\frac{a^2}{2}+\frac{3a}{8}+\frac{3}{8}\right)$

Case(I): From (3.1.8) we obtain

$$M_0^{(1)}\left(\Omega_0^{(1)}(a_P)+a_P\gamma_1^{(1)}\right)+M_1^{(1)}\left(\Omega_1^{(1)}(a_P)+a_P\gamma_2^{(1)}\right)+M_2^{(1)}\left(\Omega_2^{(1)}(a_P)+a_P\gamma_3^{(1)}\right)+M_3^{(1)}\left(\Omega_3^{(1)}(a_P)+a_P\gamma_4^{(1)}\right)$$
$$+M_4^{(1)}\left(\Omega_4^{(1)}(a_P)+a_P\gamma_5^{(1)}\right)+M_5^{(1)}\left(\Omega_5^{(1)}(a_P)+a_P\gamma_6^{(1)}\right)=a_P+a_P^3,\ (P=1,2,\dots,n+1)$$

or

$$\pi\left[M_1^{(1)}\left(\frac{a_P}{2}+1\right)+M_2^{(1)}a_P+M_3^{(1)}\left(a_P^2+\frac{3}{8}a_P+\frac{1}{2}\right)+M_4^{(1)}\left(a_P^3+\frac{1}{2}a_P\right)\right.$$
$$\left.+M_5^{(1)}\left(a_P^4+\frac{1}{2}a_P^2+\frac{5}{16}a_P+\frac{3}{8}\right)\right]=a_P+a_P^3$$

$$(P=1,2,\dots,6) \qquad (3.1.11)$$

Solution for System (3.1.9) utilizing the zeros x_k of polynomial Chebyshev $T_{n+1}(a)$ the following is how the values of the constants are determined:

$$\left[M_1^{(1)}=M_3^{(1)}=M_5^{(1)}=0, M_2^{(1)}=\frac{1}{2\pi}, M_4^{(1)}=\frac{1}{\pi}\right.$$

and $M_0^{(1)}$ is an arbitrary constant.

Substituting from the values of the constants (3.1.9) into (3.1.8) we obtain the approximate solution which is given by:

$$\varphi(a)=\frac{1}{\sqrt{1-a^2}}\left(M_0^{(1)}+\frac{1}{\pi}M^4+\frac{1}{2\pi}a^2\right)$$

or

$$\varphi(a)=\frac{M_0^{(1)}}{\sqrt{1-a^2}}+\frac{1}{\pi\sqrt{1-a^2}}\left(a^4+\frac{1}{2}a^2\right) \qquad (3.1.12)$$

Comparing the exact solution (3.1.2) with the approximate solution (3.1.10) it is clear that the approximate solution is identical to the precise answer if we take $M_0^{(1)} = A - \frac{5}{8\pi}$.

Case (II) : Similarly as in case (I) The constants' values can be found as follows:

$\left[M_4^{(2)} = 0, M_2^{(2)} = -\frac{1}{\pi}, M_0^{(2)} = -\frac{3}{2\pi}, M_3^{(2)} = -M_5^{(2)}, M_1^{(2)} = \frac{3}{16}M_5^{(2)}\right.$ and $M_5^{(2)}$ is an arbitrary constant].

If $M_5^{(2)}$ takes value close to zero, then the error given by the following table.

Table 4 . Comparison Between Exact and Approximate Solutions with the Corresponding Errors

a	φ_{approx}	φ_{exact}	Error
0.98	-0.155848649	-0.155848791	1.42×10^{-7}
0.95	-0.238789689	-0.238789832	1.43×10^{-7}
0.9	-0.32050796	-0.320508024	6.40×10^{-8}
0.75	-0.434243428	-0.434243286	-1.42×10^{-7}
0.5	-0.482412783	-0.482412783	0.0000000000
0.1	-0.478238564	-0.47823865	8.60×10^{-8}
0	-0.318309886	-0.318309886	0.0000000000
-0.1	-0.478238736	-0.47823865	-8.60×10^{-8}
-0.5	-0.482412783	-0.482412783	0.0000000000
-0.75	-0.434243145	-0.434243286	1.41×10^{-7}
-0.9	-0.320508088	-0.320508024	-6.40×10^{-8}

Case (III): As in case (I) The constants' values can be found as follows:

$$\left[M_4^{(3)} = M_5^{(3)} = 0\ ,\ M_2^{(3)} = M_3^{(3)} = -\frac{1}{\pi} \text{ and } M_0^{(3)} = M_1^{(3)} = -\frac{3}{2\pi}\right]$$

and the precise answer, which is provided by (3.1.4), is identical to the approximate solution.

Case (IV) : The constants' values are simple to determine as:

$$\left[M_4^{(4)} = M_5^{(4)} = 0, M_2^{(4)} = -\frac{1}{\pi}, M_3^{(4)} = \frac{1}{\pi}, M_0^{(4)} = -\frac{3}{2\pi} \text{ and } M_1^{(4)} = \frac{3}{2\pi}\right]$$

and the approximate solution is the same as the exact solution which is given by (3.1.5).

3.2. Application on Fredholm Integral Equations

Consider the following Fredholm integral equation:

$$\phi(a) = \sin\frac{\pi}{2}a + \int_{-1}^{1} \sin\frac{\pi}{2}a\cos\frac{\pi}{2}t\phi(c)\,dc\,,\ -1 \le a \le 1 \qquad (3.2.1)$$

using the precise answer $\phi(a) = \sin\frac{\pi}{2}a$.

For $n = 3$ the collocation points in the interval $[-1,1]$ are found from (2.3.5) as

$$a_0 = -1,\ a_1 = -\frac{1}{3},\ a_2 = \frac{1}{3},\ a_3 = 1$$

and

$$T_n = \begin{pmatrix} 1+\frac{4}{\pi} & -1 & 1+\frac{4}{\pi}-\frac{64}{\pi^3} & -1 \\ 1+\frac{2}{\pi} & -\frac{1}{3} & -\frac{7}{9}+\frac{2}{\pi}-\frac{32}{\pi^3} & \frac{23}{27} \\ 1-\frac{2}{\pi} & \frac{1}{3} & -\frac{7}{9}-\frac{2}{\pi}+\frac{32}{\pi^3} & -\frac{23}{27} \\ 1-\frac{4}{\pi} & 1 & 1-\frac{4}{\pi}+\frac{64}{\pi^3} & 1 \end{pmatrix},\ R_n = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Substituting from these matrices into (2.3.6) and solving the matrix equation $T_nX = R_n$ we obtain the inverse A_n^{-1} as follows:

$$T_n^{-1} = \begin{pmatrix} \frac{7}{32} & \frac{9}{32} & \frac{9}{32} & \frac{7}{32} \\ -\frac{1}{64}\frac{23\pi^3-146\pi^2+1314}{\pi^3} & -\frac{9}{64}\frac{3\pi^3-146}{\pi^3} & \frac{9}{64}\frac{23\pi^3+146}{\pi^3} & \frac{1}{64}\frac{23\pi^3+146\pi^2-1314}{\pi^3} \\ \frac{9}{32} & -\frac{9}{32} & -\frac{9}{32} & \frac{9}{32} \\ -\frac{9}{64}\frac{\pi^3+2\pi^2-18}{\pi^3} & \frac{27}{64}\frac{\pi^3-6}{\pi^3} & -\frac{27}{64}\frac{\pi^3+6}{\pi^3} & \frac{9}{64}\frac{\pi^3-2\pi^2+18}{\pi^3} \end{pmatrix}$$

and the coefficients $\left[\alpha_0^{(1)} = 0, \alpha_1^{(1)} = 1.140625000, \alpha_2^{(1)} = 0\right.$ and $\left.\alpha_3^{(1)} = -0.1406250002\right]$ in case of Chebyshev polynomial of the first kind $N_n(a)$, then substituting from the values $\alpha_i^{(1)}$ into (2.3.2) we obtain the approximate solution which is given by $\phi_3(a) = 1.562500001x - 0.5625000008x^3$,

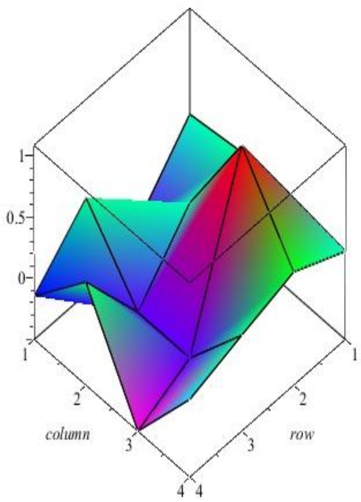

Figure 1. The matrix T_n^{-1} of the example

4. Conclusion

We presented a numerical example of a single integral equation using the Chebyshev polynomial. We note that the approximate solution is the same as the precise resolution in the second and third instances, However, in the first and second cases, the precise and approximate solutions differ. We present some numerical examples of Friedholm's integral equations using polynomials Chebyshev of the first type Tn(a), the second type Un(a), the third type Vn(a), and the fourth type Wn(a). We find that the approximate solution for all cases From the Chebyshev polynomial in each example has the same method.

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