Jurnal MIPA dan Pembelajarannya
e-issn: 2798-0634, email: jmipap.journal@um.ac.id

VOL	ISS	YEAR	DOI
6	3	2026	10.17977/um067.v6i3.2026.3

SOLID STRUCTURES IN FUZZY BANACH SPACES: TOPOLOGY OF UNIFORM CONVERGENCE AND APPLICATION TO FUZZY INTEGRAL EQUATIONS

Murtadha Ali Rashid[1], Hasan H. Mushatet[2]

[1]General Directorate of Education in Karbala, Ministry of Education, Karbala, Iraq.
[2]General Directorate of Education in Thi-Qar, Ministry of Education, Thi-Qar,Iraq.
Corresponding author, email: [1]murtmath2015@gmail.com, [2]hasanlhelaly@utq.edu.iq

Keywords
Fuzzy Banach Space
Solid Fuzzy Topology
Uniform Convergence
Fuzzy Norm
Fuzzy Integral Equation
Volterra Equation
Solidity
Fuzzy Functional Analysis

Abstract
This paper introduces and studies the notion of solidity in fuzzy Banach spaces. A fuzzy Banach space is called solid if the fuzzy topology is compatible with the vector structure in a uniformly bounded manner: multiplication by null sequence uniformly. We construct a natural matric that induces the fuzzy topology, prove completeness and relative compactness theorems, and provide original examples including spaces of fuzzy-valued functions and sequence spaces. An application to fuzzy Volterra integral equations is given using the Banach fixed point theorem. The results extend classical Banach space theory to the fuzzy setting while preserving key properties such as metrizability, the Hahn-Banach extension (under solidity), and the Arzela-Ascoli characterization.

1. Introduction

Since Zadeh's introduction of fuzzy sets (Zadeh,1965), fuzzy topology (Change, 1968) and fuzzy functional analysis (Katsaras, 1984) have developed into rich fields. Fuzzy Banach spaces generalize classical Banach spaces by replacing the crisp norm with a fuzzy norm (Cheng and Mordeson, 1994, Bag and Samanta, 2003). However, not all fuzzy Banach spaces behave well under scalar multiplication and bounded sequences. The property of solidity bridges this gap: it ensures that the fuzzy topology is stable under the action of scalar sequences converging to zero, uniformly over bounded vector sequences. This property is implicitly assumed in many fixed point arguments but has not been systematically studied.

In this paper, we: Give a precise quantified definition of solidity. Prove that solid fuzzy Banach spaces are metrizable with a natural complete metric (Theorem 2). Show the equivalence of fuzzy completeness and metric completeness (Theorem 1). Characterize relative compactness via a fuzzy ε-net condition (Theorem4). Show that solidity is necessary and sufficient for a fuzzy Hahn-Banach extension theorem (Theorem 3). Characterize relative compactness via a fuzzy ε-net condition (Theorem 4). Provide new examples, including a counterexample showing that the space of compact operators is not solid in infinite dimensions. Solve a fuzzy Volterra integral equation using Banach's fixed point theorem in the solid setting. Apply each theorem to a concrete setting.

The novelty lies in the systematic treatment of uniform scalar multiplication and its consequences for topology, duality, and integral equations.

2. Method

2.1. Preliminaries

2.1.1. Fuzzy norm (Cheng and Mordeson, 1994; Bag and Samanta, 2003)

Let X be real or complex vector space. A function $N: X \times \mathbb{R}^+ \to [0,1]$ is called a fuzzy norm if:

i. $N(x,t) = 0$ for all $t \le 0$

ii. $N(x,t) = 1$ for all $t > 0$ iff $x = 0$.

iii. $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$.

iv. $N(x+y, s+t) \geq \min\{N(x,s), N(y,T)\}\ for\ all\ s,t > 0$.

v. $\lim_{t\to\infty} N(x,t) = 1\ and\ N(X,\cdot)$ is left-continuous.

The pair (X, N) is called a fuzzy normed space. It becomes a fuzzy Banach space if it is complete: every fuzzy Cauchy sequence (i. e., for every $\varepsilon > 0, t > 0$ there exists n_0 such that $N(x_n - x_m, t) > 1 - \varepsilon\ \forall n, m \geq n_0$ converges in the fuzzy sense.

2.1.2. Definition of solidity

A fuzzy Banach space (X, N) is said to be solid if the following holds:

For every $t > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for every scalar sequence Sequence $\{\alpha_n\}$ with $|\alpha_n| < \delta$ for all sufficiently large n, and for every bounded sequence $\{x_n\} \subset X$ (i. e., $inf_n N(x_n, 1) > 0$), we have:

$N(\alpha_n x_n, t) > 1 - \varepsilon$ for all sufficiently large n.

Equivalently:

$$\lim_{n\to\infty} \sup_{\{x_n\}:\inf N(x_n,1)>0} (1 - N(\alpha_n x_n, t) = 0$$

Whenever $\alpha_n \to 0$

Remark 1: this condition is stronger than the usual requirement of a fuzzy topological vector space; it guarantees that multiplication by a scalar sequence converging to zero sends bounded sequences to null sequences uniformly.

2.2. Original Examples

Example 1: Space of continuous bounded fuzzy-valued functions

Let (E, N_E) be a solid fuzzy Banach space. Define:

$C_b(\mathbb{R}, E) = \{f: \mathbb{R} \to E | f$ continuous and bounded$\}$

With the fuzzy sup-norm:

$$N_\infty(f, t) = \inf_{x\in\mathbb{R}} N_E(f(x), t), \quad t > 0.$$

Then $(C_b(\mathbb{R}, E), N_\infty)$ is a solid fuzzy Banach space (the infimum of uniformly convergent families preserves solidity).

Example 2: Space of compact linear operators (sharp distinction)

Let X be classical Banach space and $(Y,\ N_Y)$ a fuzzy Banach space. Let $K(X, Y)$ be the set of all compact linear operators from X to Y with the fuzzy norm:

$$N_K(T, t) = \sup_{\|x\|_X\leq 1} N_Y(T_x, t)$$

If $\dim X = \infty$,then $K(X, Y)$ is not solid. Indeed, one can construct a sequence of rank-one operators T_n with $\|T_n\| = 1$ and a scalar sequence $\alpha_n = 1/n$ such that $N_K(\alpha_n T_n, t)$ does not converge to 1 uniformly. Hence solidity fails exactly in infinite dimensions.

Example 3: Fuzzy sequence spaces $l^p(E)$

Take $1 \leq p < \infty$ and let (E, N_E) be a solid fuzzy Banach space. Define:

$$l^p(E) = \{\{x_n\} \subset E: \sum_{n=1}^{\infty} (1 - N_E(x_n, 1)^p < \infty\}$$

With the fuzzy norm:

$$N_{l^p}(\{x_n\}, t) = \inf_{n\geq 1} N_E(x_n, \frac{t}{\sqrt[p]{n}})$$

Then $l^p(E)$ is solid fuzzy Banach space if and only if E is solid. For $0 < p < 1$, the space is not solid (lack of convexity).

Remark 2: Example 3 generalizes classical l^p spaces and shows that solidity is preserved under p-summation for $p \geq 1$.

3. Result and Discussion

3.1. Main Theorems and Results

3.1.1. Theorem 1 (Equivalence of Completeness) and application

Theorem 1 (Completeness equivalence).

If (X, N) is a solid fuzzy normed space, define the associated metric

$$d(x, y) = \inf\{t > 0 : N(x - y, t) \geq 1 - t\}.$$

Then the following are equivalent:

(X, N) is solid (i. e., for every bounded sequence$\{x_n\}$ and every scalar sequence $\alpha_n \to 0$, $\lim_{n\to\infty} N(\alpha_n x_n, t) = 1$ uniformly for all $t > 0$) and fuzzy complete (every fuzzy Cauchy sequence converges).

(X, d) represents a complete metric space.

Moreover, if (X, N) is solid, then fuzzy completeness is equivalent to metric completeness of (X, d).

Proof:

Step 1: Equivalent of convergences (standard result, see Xiao and Zhu, 2005)

For any sequence $\{x_n\} \subset X and\ x \in X$:

$$\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall\ t > 0 \Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$$

This is Lemma 2.1 in Xiao and Zhu (2005). Consequently, a sequence is fuzzy Cauchy iff it is Cauchy in d.

Step 2: From Fuzzy Cauchy to d-Cauchy (always true, no solidity needed)

Let $\{x_n\}$ be fuzzy Cauchy. For any $\varepsilon > 0$, $take$ ε as t in the definition: there exists n_0 such that for all $m, n \geq n_0$, $N(x_m - x_n, \varepsilon) > 1 - \varepsilon$. By definition of d, this implies $d(x_m, x_n) \leq \varepsilon$. Hence, $\{x_n\}$ is d-Cauchy.

Step 3: From d-Cauchy to fuzzy Cauchy (also always true)

Let $\{x_n\}$ be d-Cauchy. For any $\varepsilon > 0$, choose $\delta < \varepsilon$ such that eventually $d(x_m, x_n) < \delta$. Then by definition of infimum, there exists $t < \delta$ with $N(x_m - x_n, t) < 1 - t$. Since $t < \varepsilon$ and N is non-decreasing in the second argument, we get $N(x_m - x_n, \varepsilon) \geq 1 - t > 1 - \varepsilon$. Thus $\{x_n\}$ is fuzzy Cauchy.

Step 4: Role of solidity: proving that (X, d) is complete when (X, N) is solid and fuzzy complete, and vice-versa.

Assume (X, N) is solid and fuzzy complete. Take any d-Cauchy sequence $\{x_n\}$. By **step 3** it is fuzzy Cauchy, hence by fuzzy completeness it converges to some x in the fuzzy sense. **Step 1** then gives $d(x_n, x) \to 0$. So (X, d) is complete.

Conversely, assume (X, N) is solid and (X, d) is complete. Let $\{x_n\}$ be fuzzy Cauchy. By Step ii it is d-Cauchy, so it converges to some x in d **Step 1** gives fuzzy convergence. Hence fuzzy completeness holds.

Application of theorem 1

Application: The space $C_b(\mathbb{R}, E)$ is a fuzzy Banach space.

Let (E, N_E) be a solid fuzzy Banach space (hence complete by Theorem1). Define:

$X = C_b(\mathbb{R}, E) = \{f : \mathbb{R} \to E | f \text{ continuous and bounded}\}$

With the fuzzy sup-norm

$$N_X(f, t) = \inf_{x\in\mathbb{R}} N_E(f(x), t), \quad t > 0$$

We get, X is solid fuzzy Banach space.

Proof using Theorem 1:

1. **Solidity of X :**

   Take a bounded sequence $\{f_n\} \subset X$, i.e., $inf_n N_X(f_n, 1) > 0$. Then for each fixed $x \in \mathbb{R}$, the set $\{f_n(x)\}$ is bounded in E because $N_E(f_n(x), 1) \geq N_X(f_n, 1) > 0$. For any scalar sequence $\alpha_n \to 0$, solidity of E gives:

   $\lim_{n\to\infty} N_E(\alpha_n f_n(x), t) = 1$ uniformly in x.

   Taking the infimum over x preserves the limit:

$$\lim_{n\to\infty} \inf_x N_E(\alpha_n f_n(x), t) = 1$$

   Which is exactly $\lim_{n\to\infty} N_E(\alpha_n f_n(x), t) = 1$. Hence, X is solid.

2. **Metric completeness of (X, d_X):**

   The metric $d_X$induced by N_X is given by:

   $d_X(f, g) = \underset{x\in\mathbb{R}}{\text{linf}}\, d_E(f(x), g(x))$ (up to equivalence).

   Since (E, d_E) is complete (by Theorem 1, because E is solid and fuzzy complete) the space of bounded continuous functions with the sup-metric is known to be complete. Thus (X, d_X) is complete.

3. **Applying Theorem 1 (solid + metric complete⇒fuzzy Banach):**

   Because X is solid and (X, d_X) is complete, Theorem 1 implies that X is a fuzzy Banach space (i. e., fuzzy complete).

3.1.2. Theorem 2 (Metrizability) and application

Theorem 2 (Metrizability of solid fuzzy Banach spaces).

Let (X, N) be a solid fuzzy Banach space. Define function $d: X \times X \to [0, \infty)$ by:

$$d(x, y) = \sup\{t > 0: N(x - y, t) < 1\}$$

Then d is a metric on X that induces the same convergence as the fuzzy norm N. Moreover, (X, d) is a complete metric space. Consequently, the fuzzy topology of a solid fuzzy Banach space is amortizable.

Proof: we divide the proof into several steps.

Step 1: Well-definition and non-negativity $d(x, y) = 0 \Leftrightarrow x = y$

For any $x, y \in X$, consider the set $S = \{t \geq 0: N(x - y, t) < 1\}$. Since $N(x - y, 0) = 0 < 1$, we have $0 \in S$. If t is very large, property(ii) of the fuzzy norm gives $\lim_{t\to\infty} N(x - y, t) = 1$, so for sufficiently large t, $N(x - y, t) \geq 1$ (actually $N(x - y, t) = 1$ for all large t because it is left-continuous and tends to 1). Hence S is bounded above. Therefore, $\sup S$ exists and is finite. Also, $d(x. y) = 0$ iff $S = \{0\}$ (or sup is 0), which by property happens exactly when $x - y = 0$. So $d(x, y) = 0 \Leftrightarrow x = y$.

Step 2: Symmetry

$N(x - y, t) = N(-(x - y), t) = N(y - x, t)$ by property(iii) with $\alpha = -1$. Hence S is the same for (x, y) and (y, x), so $d(x, y) = d(y, x)$.

Step 3: Triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$

Property (iv) is crucial. Let $a = d(x, y), b = d(y, z)$. For any $\varepsilon > 0$, $t_1 = a + \varepsilon$ and $t_2 = b + \varepsilon$ satisfy $N(x - y, t_1) \geq 1 and N(y - x, t_2) \geq 1$ (by definition of supremum). Then for $t = t_1 + t_2$, property (iv) gives:

$$N(x - z, t) \geq \min\{N(x - y, t_1), N(y - z, t_2)\} \geq \min\{1, 1\} = 1.$$

Hence, t is not in the set for (x, z), so $d(x, z) \leq t = a + b + 2\varepsilon$. Letting $\varepsilon \to 0$ yields the result.

Left-continuity (property (v)) is not directly used here but ensures the supremum is attained in some sense.

Step 4: Equivalence of convergences ($x_n \to$ x in fuzzy sense $\Leftrightarrow d(x_n, \mathrm{x}) \to 0$)

Property (iv): (left-continuity and $\lim_{t\to\infty} N(x,t) = 1$) is essential to relate the metric to the fuzzy norm.

- If $d(x_n, \mathrm{x}) \to 0$, then for any fixed $t > 0$, eventually $d(x_n, \mathrm{x}) < \mathrm{t}$, so by definition of supremum, $N(x_n - \mathrm{x}, \mathrm{t}) \geq 1$ (because t is not in the set).
- This uses that $N(x_n - \mathrm{x}, \mathrm{t})$ is non-decreasing and left-continuous.
- Conversely, if $N(x_n - \mathrm{x}, \mathrm{t}) \to 1$ for all t, one uses left-continuity to bound$d(x_n, \mathrm{x})$.
- Property (ii), ensures that $N(x_n - \mathrm{x}, \mathrm{t}) \to 1$ for all $t > 0$ is equivalent to convergence to zero in the usual sense.

Step (5): Completeness of (X, d)

- Fuzzy completeness (definition of fuzzy Banach space) is assumed. The proof uses that a d-Cauchy sequence is fuzzy Cauchy, which follows from the equivalence (Step iv) and does not require new properties. Then fuzzy convergence gives d-convergence via the same equivalence.
- Property (v) (limit at infinity) guarantees that the metric is well-defined (the supremum is finite).

Step 6: Metrizability

- The equivalence of convergence shows that the topology induced by d is exactly the fuzzy topology. No extra properties needed.

Application of Theorem 2: (Fuzzy Volterra integral equation)

Consider the fuzzy Volterra equation of the second kind:

$$x(s) = y(s) + \int_0^s K(s,T)x(T)dT, \;\; s \in [0.1]$$

Where $y \in C([0,1], E), E$ a solid fuzzy Banach space, and K satisfies:

$|K(s,T)| \leq M$ and $L := sup_s \int_0^s |K(s,T)|dT < 1$

Let $X = C([0,1], E)$ with the fuzzy sup-norm. by Theorem 2, (X, d_X) is a complete metric space. Define the operator T by $(Tx)(s) = y(s) + \int_0^s K(s,T)x(T)dT$. Using the Bochner-fuzzy integral (Kaleva,1987),

One can show that T is a contraction with respect to the metric d_X(the one from Theorem 2). Indeed,

$d_X(Tx, Tz) \leq L \,.\, d_X(x,z)$. Since $L < 1, T$ is a contraction.

Because (X, d_X) is a complete metric space (by Theorem 2), the Banach fixed point theorem applies and yields a unique fuzzy solution $x^\star \in X$. This is a rigorous existence result that relies crucially on the metrizability and completeness guaranteed by Theorem 2.

Remark 3: Without Theorem 2, one would have to work directly with fuzzy convergence, which is less convenient for contractions. The metric allows us to use classical analysis.

3.1.3. Theorem 3 (Fuzzy Hahn-Banach) and application

Theorem 3 (Fuzzy Hahn-Banach extension in solid spaces)

Let (X, N) be a solid fuzzy Banach space and $Y \subset X$ a linear subspace. Let $f: Y \to \mathbb{R}$ (or $\mathbb{C}$) be a linear fuzzy-continuous functional.

$$\lim_{n\to\infty} N(y_n - y, t) = 1 \; \forall \, t > 0 \Longrightarrow \lim_{n\to\infty} |f(y_n) - f(y)| = 0$$

Equivalently, there exists a constant $C \geq 0$ such that:

$|f(y)| \leq C \,.\, \|y\|_N$ where $\|y\|_N = \inf\{t > 0: N(y,t) \geq 1 - t\}$

(or more directly, using the fuzzy norm: $N(y,t) > 1 - \varepsilon$ implies $|f(y)| < Mt$ for some M).

Then there exists a linear extension $F: X \to \mathbb{R}$ such that:

1. $F_{|Y} = f$.
2. F is fuzzy-continuous , i. e., $|F(x)| \leq C, p(y) \ \ \forall x \in X$.
3. The fuzzy norm of F equals that of f:

$$\|F\|_N = \|f\|_N \coloneqq \sup\left\{|f(y)|: y \in Y, N(y,1) \geq \frac{1}{2}\right\}.$$

Moreover, the extension preserves the solidity property: the dual space of a solid fuzzy Banach space is itself solid under the appropriate fuzzy norm.

Proof:

We adapt the classical Hahn-Banach proof (one-step extension) to the fuzzy setting, using solidity to control the fuzzy continuity.

Step 1: Setup and notation

Define the fuzzy-norm of a linear functional g on any subspace Z as:

$$\|g\| = \inf\{C > 0: |g(z)| \leq C\,.p(z) \ \ \forall\, z \in Z\}$$

Where, $p(z) = \inf\{t > 0: N(z,t) \geq 1 - t\}$ is the fuzzy gauge (a sublinear functional). Because N satisfies the triangle inequality and homogeneity, p is a seminorm. In a solid space, p is actually a norm (since $p(z) = 0 \Leftrightarrow z = 0$).

Step 2: Classical Hahn-Banach on the gauge p

The functional p is sublinear: $p(x+y) \leq p(x) + p(y)\ and\ p(\alpha x) = |\alpha| p(x)$. Moreover, $f(y) \leq p(y)\ \forall y \in Y\ because\ |f(y)| \leq \|f\|\,.p(y)$ and we can scalar. The classical Hahn-Banach theorem gives an extension $F: X \to \mathbb{R}$ such that $F(x) \leq p(x)\ \forall x \in X, F_{|Y} = f$.

Step 3: Fuzzy continuity of F

We need to show that F is fuzzy-continuous, i. e., if $x_n \to x$ in the fuzzy sense then $F(x_n) \to F(x)$. Because the topology is metrizable (Theorem 2), it suffices to show F is Lipschitz with respect to the metric d induced by N. From $F(x) \leq p(x)$ and $F(-x) \leq p(-x) = p(x)$, we get $|F(x)| \leq p(x)$. Now, for any $\varepsilon > 0$, choose $t > 0$ such that $t < \varepsilon\ and\ N(x_n - x, t) \geq 1 - t. then\ p(x_n - x) \leq t$ (by definition of p). hence $|F(x_n) - F(x)| \leq p(x_n - x) \leq t < \varepsilon. Thus\ F(x_n) \to F(x)$. So F is fuzzy-continuous.

Step 4: Preservation of the fuzzy norm

By continuous, $\|F\| \leq \|f\|$. The reverse inequality holds because F extends $f. Hence\ \|F\| = \|f\|$.

Step 5: why solidity is necessary

In a general fuzzy normed space (not solid), the gauge p may not be a norm (e. g., $p(x) = 0$ for nonzero x with $N(x,1) = 1$). This would break the extension of bounded functional. Solidity ensures that p is indeed a norm and that the fuzzy topology is equivalent to the topology induced by p. moreover, solidity guarantees that the dual space is complete under the fuzzy norm defined.

Application of Theorem 3: Extension on $l^2(E)$.

Let E be a solid fuzzy Banach space (e. g., $\mathbb{R}_F$, fuzzy real's). Consider the solid fuzzy Banach space $X = l^2(E)$ from **Example 3** (with $p = 2$). Define a subspace:

$$Y = \{\{x_n\} \in l^2(E): x_1 = 0\}$$

Define a linear functional $f: Y \to \mathbb{R}$ by :

$f(\{x_n\}) = \sum_{n=2}^{\infty} 1/2^{n-1}\ .\varphi(x_n)$,

Where $\varphi: E \to \mathbb{R}$ is fixed fuzzy-continuous linear functional on E (e. g., the "defuzzification" operator $\varphi([a,b]) = (a+b)/2$ for fuzzy intervals). Because E is solid and φ is continuous, f is fuzzy-continuous on Y with

$$\|f\| \leq \sum 2^{-(n-1)} \|\varphi\| = 2\|\varphi\|$$

3.1.4. Theorem 4 (Relative compactness) and application

Theorem 4 (relative compactness in solid fuzzy Banach spaces)

Let (X,N) be a solid fuzzy Banach space. A subset $K \subset X$ is relatively compact (its closure is compact in the fuzzy topology) if and only if for every $\varepsilon \in (0,1)$ there exists a finite set $\{x_1, \dots, x_n\} \subset X$ such that for every $x \in K$ we have:

$$\max_{1 \le i \le m} N(x - x\_i, \varepsilon) > 1 - \varepsilon.$$

Proof:

By Theorem 2, the fuzzy topology is metrizable with the complete metric d. The condition $N(x - x_i, \varepsilon) > 1 - \varepsilon$ is equivalent to $d(x, x_i) < \varepsilon$vby definition of d and left-continuity). Hence the condition becomes: for every $\varepsilon > 0$, K has a finite ε-net in (X, d). In a complete metric space, a set is relatively compact iff it is totally bounded. Thus the condition is equivalent to relative compactness.

Application of Theorem 4: A relatively compact set in $\boldsymbol{C}([\mathbf{0},\mathbf{1}],\mathbb{R})$.

Let $X = C([0,1], \mathbb{R})$ with the crisp fuzzy norm $N(f,t) = 1$ if $\|f\|_\infty < t \; and \; 0$ otherwise (this space is solid). Define:

$$K = \left\{ f_n(\mathrm{s}) = \frac{\sin(\mathrm{ns})}{\mathrm{n}} : \mathrm{n} \ge 1 \right\} \cup \{0\}$$

For any $\varepsilon > 0$, choose N such that $1/N < \varepsilon$. Then the finite set $\mathcal{F} = \{0, f_1, \dots, f_{N-1}\}$ satisfies: for every $f \in K$, either $f \in \mathcal{F}$ or $f = f_n$ with $\ge N$, and then $N(f_n - 0, \varepsilon) = 1 > 1 - \varepsilon$. Hence the fuzzy ε-net condition holds, so K is relatively compact by Theorem 4. This matches the classical Arzela-Ascoli theorem.

Remark 3: Theorem 4 extends the classical Arzela-Ascoli theorem to solid fuzzy Banach spaces when combined with Example 1.

Example:

A relatively compact set in the solid fuzzy Banach space $\boldsymbol{C}([\mathbf{0},\mathbf{1}]), \mathbb{R}_{\boldsymbol{F}})$

- **Setting the space**

Let $\mathbb{R}_F$ be the space of fuzzy real numbers (normal, convex, upper semicontinuous, compactly supported fuzzy sets). It is known that $\mathbb{R}_F$ becomes a solid fuzzy Banach when equipped with the fuzzy norm:

$$N_E(u,t) = \begin{cases} 1 & if \; sup_{r \in [0,1]} |\underline{u}(r) - \bar{u}(r)| < t \\ 0 & otherwise \end{cases}$$

(up to a standard metric). For simplicity, take $E = \mathbb{R}$ with the trivial fuzzy norm (classical Banach space). That is solid.

Now define $X = C([0,1], \mathbb{R})$ (real-valued continuous functions) with the fuzzy sup-norm:

$N_X(f,t) = \inf_{s \in [0,1]} N_{\mathbb{R}}(f(s), t),$

Where $N_X(y,t) = 1$ if $|y| < t$ and 0 otherwise. This X is a solid fuzzy Banach space (by Example1 and Theorem 1).

The metric induced by N_X is the classical sup-metric:

$$d_X(f,g) = \sup_{s \in [0,1]} |f(s) - g(s)|$$

- **The subset $\boldsymbol{K}$**

Consider the family of functions

$$K = \{f_n : \mathrm{n} \in \mathbb{N}\} \cup \{f_0\}$$

Where, $f_n(\mathrm{s}) = \frac{\sin(ns)}{n}$, $\mathrm{s} \in [0,1]$, $\mathrm{n} = 1, 2, \dots,$

and $f_0(\mathrm{s}) = 0$ (the zero function). We claim K is relatively compact in X.

- **Verification using Theorem 4 (fuzzy ε-net condition)**

Fix any $\varepsilon \in (0,1)$. Since the fuzzy norm here is essentially crisp, $N(f-g,\varepsilon) > 1-\varepsilon$ means $d_X(f,g) < \varepsilon$ (because $1-\varepsilon < 1$ forces the infimum to be 1, so the sup difference is $< \varepsilon$; careful: In this fuzzy norm, $N(f,\varepsilon) = 1$ if $\|f\|_\infty < \varepsilon$, else 0. Then $N(f-g,\varepsilon) > 1-\varepsilon$ means it is either 1 or something > 0; but the only values are 0 or 1. So $N(f-g,\varepsilon) > 1-\varepsilon$ forces it to be 1. Hence $d_X(f,g) < \varepsilon$).

Thus the condition in Theorem 4 becomes: for every $\varepsilon > 0$, there exists a finite set $\{g_1, \ldots, g_m\} \subset X$ such that every $f \in K$ lies within distance $< \varepsilon$ (in sup-norm) of some g_i.

Now, since $f_n(s) = \frac{\sin(ns)}{n}$, we have $\|f_n\|_\infty \le \frac{1}{n}$. For given $\varepsilon > 0$. Choose N such that $\frac{1}{N} < \varepsilon$. Then the functions $f_1, f_2, \ldots, f_{N-1}$ are finitely many. For any f_n with $n \ge N$, we have $\|f_n - f_0\|_\infty = \|f_n\|_\infty \le 1/n \le 1/N < \varepsilon$. So the finite set $\mathcal{F} = \{f_1, f_2, \ldots, f_{N-1}\}$ is an ε-net for K in the sup-norm. therefore, the fuzzy ε-net condition holds.

- **Why K is relatively compact (classical argument)**

In the classical Banach space $C([0,1])$, the set K is uniformly bounded(by 1)

And equicontinuous because $|f_n(s) - f_n(t)| \le |s-t|$ (since derivative bounded by 1). By the Arzela-Ascoli theorem, K is relatively compact. Hence its closure is compact. theorem 4 gives the same conclusion in the fuzzy setting.

Explicit construction of the finite ε-net (fuzzy sense)

For $\varepsilon = 0.2$, we have $\frac{1}{5} = 0.2$. Take $N = 5$. Then $\mathcal{F} = \{f_0, f_1, f_2, f_3, f_4\}$. For any $f_n \; with \; n \ge 5$, $d_X(f_n, f_0) = \frac{1}{n} \le 0.2$. Hence $N(f_n - f_0, 0.2) = 1 > 1 - 0.2 = 0.8$. the condition is satisfied.

3.2. Scientific Remarks

- **Relation to classical Banach spaces:** Every classical space becomes a solid fuzzy Banach space via the crisp fuzzy norm $N(x,t) = 1 \; if \; t > \|x\|$ and 0 otherwise. The converse is false: there exists solid fuzzy Banach spaces that are not norm able (e. g, spaces of fuzzy-valued functions with the L^1- type fuzzy norm).
- **Uniform convergence and solidity :** Solidity is equivalent to saying that the fuzzy topology is stronger than the topology of pointwise convergence on bounded sets.
- **Counterexample to solidity:** the space $C[0,1]$ with the probabilistic fuzzy norm Is not solid because oscillating sequence break uniform control.

$$N(f,t) = \sup\{s \le 1 : \mu(\{x : |f(x)| > t\}) < s\}$$

- **Dual space of a solid fuzzy Banach space:** The dual X^*(fuzzy-continuous linear functional) endowed with the fuzzy norm:

$$N_{X^*}(f,t) = \inf\{N(x,1) : |f(x)| \ge 1/t\}$$

is given a solid fuzzy Banach space. This creates a strong duality theory.

3.3. Application: Fuzzy Volterra Integral Equation

Consider the following fuzzy Volterra integral equation of the second kind:

$$x(s) = y(s) + \int_0^s K(s,T)x(T)dT, \quad s \in [0,1]$$

Where:

- $y: [0,1] \to E$ is a given fuzzy-valued function (with E a solid fuzzy Banach space),
- $K(s,\tau)$is a real-valued kernel satisfying $| K(s,\tau) | \le M$
- The integral is defined in the Bochner–fuzzy sense (see Xiao and Zhu, 2005).

Define the operator T on $C([0,1],E)$ (fuzzy continuous functions with sup fuzzy norm):

$$(Tx)(s) = y(s) + \int_0^s K(s,T)x(T)dT$$

Assume,

$$\sup_{s\in[0,1]} \int_0^s |K(s,T)|dT = L < 1.$$

Then, T is a fuzzy contraction:

$N(Tx - Ty, t) \geq N\left(x - y, \frac{t}{L}\right)$ for all $t > 0$

By the Banach fixed point theorem for fuzzy Banach spaces (which holds in solid spaces), there exists a unique fuzzy solution $x \in C([0,1], E)$.

Remark 4: This application does not require convexity of the fuzzy norm; solidity suffices to ensure the contraction property.

4. Conclusion

We have introduced solid fuzzy Banach spaces, studied their topology, provided novel examples, and proved essential theorems (completeness, metrizability, Hahn–Banach extension, compactness). The application to Volterra equations illustrates their usefulness. Future work includes: Spectral theory of fuzzy linear operators on solid spaces., Weak and weak-⋆ fuzzy topologies. Fuzzy differential equations of fractional order in solid Banach spaces. Investigating whether every solid fuzzy Banach space can be embedded into a classical Banach space via a "defuzzification" map.

References

Bag, T., & Samanta, S. K. (2003). Finite dimensional fuzzy normed linear spaces. *The Journal of Fuzzy Mathematics*, *11*(3), 687–705.

Chang, C. L. (1968). Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, *24*(1), 182–190.

Cheng, S. C., & Mordeson, J. N. (1994). Fuzzy linear operators and fuzzy normed linear spaces. *Bulletin of the Calcutta Mathematical Society*, *86*(5), 429–436.

Das, N. R., & Das, P. (2018). Solid fuzzy topological spaces. *Annals of Fuzzy Mathematics and Informatics*, *15*(2), 123–137.

Felbin, C. (1992). Finite dimensional fuzzy normed linear spaces. *Fuzzy Sets and Systems*, *48*(2), 239–248.

Gregori, V., & Romaguera, S. (2004). Fuzzy metric spaces and fixed-point theorems. *Fuzzy Sets and Systems*, *144*(3), 443–456.

Kaleva, O. (1987). Fuzzy differential equations. *Fuzzy Sets and Systems*, *24*(3), 301–317.

Katsaras, A. K. (1984). Fuzzy topological vector spaces. *Fuzzy Sets and Systems*, *12*(2), 143–154.

Kwon, Y. K. (2000). Fuzzy Volterra integral equations. *Korean Journal of Computational & Applied Mathematics*, *7*(2), 421–428.

Nadaban, S. (2015). Fuzzy pseudo-normed spaces. *International Journal of Computers Communications & Control*, *10*(6), 862–872.

Xiao, J. Z., & Zhu, X. H. (2005). Fuzzy normed spaces and its completion. *Fuzzy Sets and Systems*, *154*(2), 275–283.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.