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COMPARISON BETWEEN THE TWO METHODS OF MAXIMUM LIKELIHOOD AND PARTIAL ESTIMATORS OF THE TOPP - LEONE DISTRIBUTION FUNCTION AND RELIABILITY USING SIMULATION

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Abstract
The study looked at how to estimate the reliability function for the Topp and Leone distribution with a shape parameter by comparing two methods: the maximum likelihood method and the method of partial estimators. The software function was used to calculate the distribution parameter estimates. An experimental study was also conducted using simulation for comparison purposes and to practically demonstrate the efficiency of these methods. This was done by relying on specific observations for different samples, comparing them based on several metrics represented by the mean squared error. It was found that as the sample size increased, the maximum likelihood estimator became the best.

1. Introduction

Many researchers have focused on conducting studies on failure times and reliability for most continuous distributions, especially non-mixed ones such as the Weibull distribution, Gamma distribution, Exponential distribution, and Normal distribution. With the continuous advancement of technology, these distributions have received wide attention due to their prominence in many fields, including the field of reliability (Burgazzi, 2003). Given that the Topp-Leone distribution is one of the failure models that examines the performance of identical and independent devices and equipment, along with its significant importance in the field of reliability, Interest and research in estimating the reliability function for this distribution have increased to determine the operational lifespan of several devices and equipment by representing them with a single function to understand the efficiency of these devices and their ability to operate for long periods (Okorie & Nadarajah, 2019).

And then evaluating these machines and equipment for future planning and development, as the Topp-Leone distribution is a continuous life distribution and one of the modern distributions that has seen increased demand recently as a probabilistic distribution for modeling life phenomena, first proposed by Topp and Leone in 1955 (Sangsanit & Bodhisuwan, 2016).

The probability density function of the Topp-Leone distribution

The probability density function (p.d.f) of the random variable (x) which follows a Topp-Leone distribution is as follows (Xie & Singh, 2013):

$$f(x,\theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \qquad (1)$$

$$0 < x < 1 \qquad , \qquad \theta > 0$$

Since $\boldsymbol{x}$ represents the random variable, and θ represents the shape parameter.

From equation (1) above, we can prove that the distribution function is a probability function as follows:

$$f(x,\theta) = \int_0^1 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx$$

$$f(x,\theta) = \theta\int_0^1 (2-2x)(2x-x^2)^{\theta-1}dx$$

$$= \theta[\frac{(2x-x^2)^{\theta}}{\theta}]_0^1$$

$$= [(2x-x^2)^{\theta}]_0^1$$

$$= [(2-1^2)^{\theta}] = 1$$

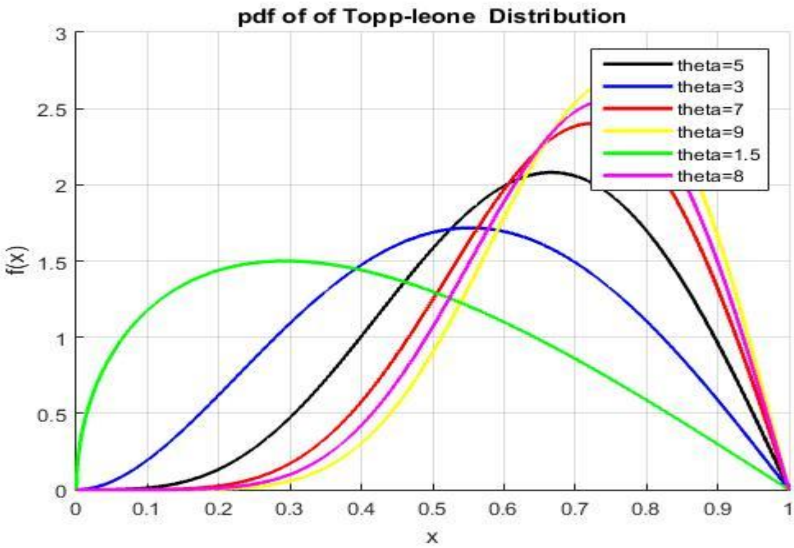

Figure 1. Shows the PDF curve for the Type I distribution

Cumulative distribution function

From equation (1) and based on the above data, we were able to prove that the distribution function is a cumulative distribution function (c.d.f) as follows (Drew et al., 2000):

$$f(x,\theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \qquad 0 < x < 1 \ , \quad \theta > 0$$

$$f(x,\theta) = \int_o^t 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx$$

$$f(x,\theta) = \int_o^t \theta(2-2x)(2x-x^2)^{\theta-1}dx$$

$$= \theta[\frac{(2x-x^2)^{\theta}}{\theta}]_0^t$$

$$= (2x-x^2)^{\theta}$$

$$F(x,\theta) \ = x^{\theta}(2-x)^{\theta} \qquad (2)$$

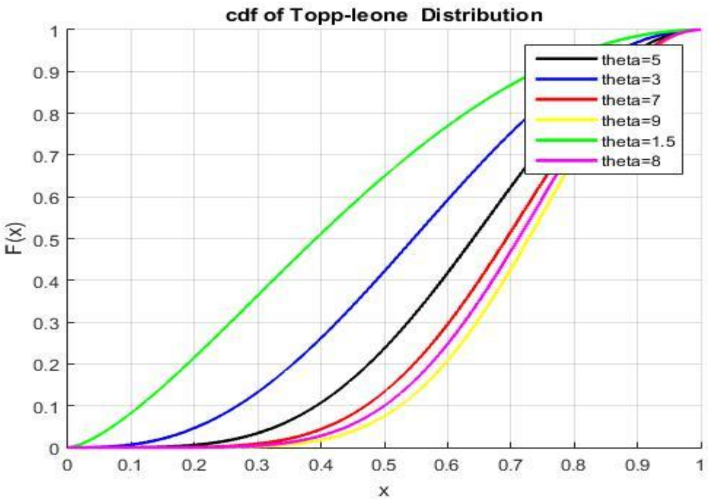

Figure 2. shows the CDF curve for the Topp-Leone distribution.

And after proving that the distribution has a probability density function and a cumulative density function, we can now find the first expectation of the distribution from equation (1) as follows (Sangsanit & Bodhisuwan, 2016):

$$f(x,\theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \qquad 0 < x < 1\ ,\theta > 0$$

$$E(x) = \int_0^1 2\theta x x^{\theta-1}(1-x)(2-x)^{\theta-1}dx$$

$$E(x) = \theta[\left.\frac{(2x-x^2)^\theta}{\theta}\right]_0^1 - \frac{1}{\theta}\int_0^1 (2x-x^2)^\theta dx]$$

$$E(x) = 1 - \int_0^1 (2x-x^2)^\theta dx = 1 - \int_0^{\frac{\pi}{2}} (1-sin^2 w)^\theta \, cos\, w \quad dw$$

$$E(x) = 1 - \int_0^{\frac{\pi}{2}} cos^{2\theta+1} w \quad dw = 1 - \int_0^{\frac{\pi}{2}} sin^\theta w \, cos^{2\theta+1} w \quad dw$$

$$E(x) = 1 - \int_0^{\frac{\pi}{2}} sin^{2n-1} w \, cos^{2m+1} w \quad dw = 1 - \frac{1}{2}\beta(n,m)$$

$$\therefore E(x) = 1 - \int_0^{\frac{\pi}{2}} cos^{2\theta+1} w \quad dw = 1 - \frac{1}{2}\beta(\frac{1}{2},\theta+1)$$

$$E(x) = 1 - \frac{1}{2}\beta\left(\frac{1}{2},\theta+1\right) = 1 - \frac{1}{2}\frac{\frac{\Gamma 1}{2} * \Gamma\theta + 1}{\Gamma\theta + 3/2}$$

$$\therefore E(x) = 1 - \frac{\sqrt{\pi}}{2}\frac{\Gamma(\theta+1)}{\Gamma\left(\theta+\frac{3}{2}\right)} \qquad (3)$$

We can also find the second expectation of the distribution as follows (Behairy et al., 2020):

$$f(x,\theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \qquad 0 < x < 1 \ , \theta > 0$$

$$E(x^2) = \int_0^1 2\theta x^2 x^{\theta-1}(1-x)(2-x)^{\theta-1}dx$$

$$E(x^2) = \theta[\left.\frac{(2x-x^2)^\theta}{\theta}\right]_0^1 - \frac{2}{\theta}\int_0^1 x(2x-x^2)^\theta dx]$$

$$E(x^2) = 1 - 2\int_0^1 x(2x-x^2)^\theta dx = 1 - 2\int_0^1 x(1-(1-x)^2)^\theta dx$$

$$= 1 - 2\int_0^1 x(2x-x^2)^\theta dx = 1 - 2\int_0^{\frac{\pi}{2}} (1-\sin w)\cos w^{2\theta} \ \cos w \quad dw$$

$$= 1 - 2\int_0^{\frac{\pi}{2}} (1-\sin w)\cos w^{2\theta+1}$$

$$= 1 - 2\int_0^{\frac{\pi}{2}} \cos w^{2\theta+1} \ dw + 2\int_0^{\frac{\pi}{2}} \sin w)\cos w^{2\theta+1}\, d\, w)$$

$$E(x^2) = \int_0^{\frac{\pi}{2}} \sin^{2n-1} w \cos^{2m+1} w \quad dw = \frac{1}{2}\beta(n,m)$$

$$E(x^2) = \int_0^{\frac{\pi}{2}} \cos^{2\theta+1} w \quad dw = \frac{1}{2}\beta(\frac{1}{2},\theta+1)$$

$$E(x^2) = 1 - \ \beta\left(\frac{1}{2},\theta+1\right) + \beta(1,,\theta+1)$$

$$E(x^2) = 1 - \frac{\frac{\Gamma 1}{2} * \Gamma(\theta+1)}{\Gamma(\theta+\frac{3}{2})} + \frac{\Gamma 1 * \Gamma(\theta+1)}{\Gamma(\theta+2)}$$

$$E(x^2) = 1 - \sqrt{\pi}\frac{\Gamma(\theta+1)}{\Gamma\left(\theta+\frac{3}{2}\right)} + \frac{1}{\theta+1} \qquad (4)$$

$$v(x) = E(x^2) - E(x)$$

$$v(x) = (1 - \sqrt{\pi}\frac{\Gamma(\theta+1)}{\Gamma(\theta+\frac{3}{2})} + \frac{1}{\theta+1}) - (1 - \frac{\sqrt{\pi}}{2}\frac{\Gamma(\theta+1)}{\Gamma(\theta+\frac{3}{2})})^2$$

$$\mathrm{v(x)} = \frac{1}{\theta+1} - \frac{\pi}{4}\frac{(\Gamma(\theta+1))^2}{(\Gamma(\theta+3/2))^2} \qquad (5)$$

Function Reliability

The reliability function R(t) is defined from both statistical and probabilistic perspectives as follows (Burgazzi, 2003):

$$R(t) = P\ (T > t) \qquad (6)$$

If the random variable T≥0 represents the accumulated time of the system's life until a failure or breakdown occurs, and it has a probability density function f(t) and a cumulative distribution function F(t), then the reliability of the device or machine at time (t) is R(t), which takes the following form:

$$R(t) = P(T > t)$$

$$R(t) = \int_{t}^{\infty} f(u)du$$

$$R(t) = 1 - \int_{-\infty}^{t} f(u)du$$

$$R(t) = 1 - \int_{0}^{t} f(u)du$$

$$R(t) = 1 - P(T < t)$$

$$R(t) = 1 - F(t)$$

Since F(t) represents the cumulative distribution function and takes the following form:

$$F(t) = P(T \leq t)$$

$$F(t) = \int_{0}^{t} f(u)du \qquad (7)$$

One of the most important characteristics of the reliability function is that it is a decreasing function over time, meaning that:

$$R(0) = 1$$

$$R(\infty) = 0$$

2. Method

Method of Estimated

There are many methods for estimating parameters, reliability functions, indicators, and related functions specific to the Topp-Leone distribution. Some of these methods will be explained in the case of censored data, and among these methods are (Brito et al., n.d.):

Maximum Likelihood Method

This method is considered one of the important and commonly used estimation methods, and the reason for this is that the Maximum Likelihood Estimation (MLE) method possesses a set of good properties, including sufficiency and sometimes consistency. The goal of this method is to find estimated values for the parameters to be estimated by maximizing the likelihood function of the random variables, denoted by the symbol (L) (Glas, 2016).

If the random variable (x) has a probability density function as in equation (1-2), then the maximum likelihood function for independent random variables is:

$$L(x_1, x_2, \dots \dots . x_n, \theta) = f\left(x_{1,}, \theta\right). f(x_2, \theta) \dots f(x_n, \theta)$$

$$\therefore L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$f(x, \theta) = 2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1}dx \qquad 0 < x < 1 \ , \theta > 0$$

$$Lf(x, \theta) = (2\theta)^n \prod x^{\theta-1} \prod (1-x) \prod (2-x)^{\theta-1} \, dx$$

$$\ln Lf(x, \theta) = n\ln(2\theta) + (\theta - 1)\sum_{i=1}^{n} \ln(x) + \sum_{i=1}^{n} \ln(1-x) + (\theta - 1)\sum_{i=1}^{n} \ln(2-x)$$

$$\ln Lf(x, \theta) = n\ln(2\theta) + \left(\theta \sum_{i=1}^{n} \ln(x) - \sum_{i=1}^{n} \ln(x)\right) + \sum_{i=1}^{n} \ln(1-x) + \left(\theta \sum_{i=1}^{n} \ln(2-x) - \sum_{i=1}^{n} \ln(2-x)\right)$$

$$\frac{\partial \ln Lf(x, \theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x) + \sum_{i=1}^{n} \ln(2-x) = 0$$

By setting the derivative to zero:

$$\frac{-n}{\theta} = \sum_{i=1}^{n} \ln(x) + \sum_{i=1}^{n} \ln(2-x)$$

$$\hat{\theta} = \frac{-n}{\left(\sum_{i=1}^{n} \ln(x) + \sum_{i=1}^{n} \ln(2-x)\right)} \qquad (8)$$

Partial Estimates Method

This method relies on the Cumulative Distribution Function (CDF), assuming that he is an estimator of the cumulative distribution function by finding the estimators that minimize the function at its endpoints, as shown below (Xun et al., 2013):

$$F(x, \theta) \ = x^{\theta}(2-x)^{\theta}$$

$$p_i = x^{\theta}(2-x)^{\theta}$$

And to simplify the equation, we take the logarithm of both sides of the equation to get the following form:

$$\ln p_i = \theta \ln x + \theta \ln(2-x)$$

By setting the derivative to zero:

$$\ln p_i - \theta \ln x - \theta \ln(2-x) = 0$$

By squaring both sides of the equation and taking the sum, the equation takes the following form:

$$\sum_{i=1}^{n} [\ln p_i - \theta \ln x - \theta \ln(2-x)]^2 = 0$$

We differentiate with respect to the parameter θ:

$$2\sum_{i=1}^{n} [\ln p_i - \theta \ln x - \theta \ln(2-x)]\,[-\ln x - \ln(2-x)] = 0$$

Divided by 2 :

$$\sum_{i=1}^{n} [\ln p_i - \theta \ln x - \theta \ln(2-x)]\,[-\ln x - \ln(2-x)] = 0$$

Knowing that xi, ordered statistics, and non-parametric estimation represent the statistics.

p_i It takes the following form:

$$P_i = \frac{i-0.3}{n+0.25}$$

We multiply the first bracket by the second and add the sum to the values:

$$\sum_{i=1}^{n} [\ln p_i - \theta \ln x - \theta \ln(2-x)]\,[-\ln x - \ln(2-x)] = 0$$

$$\sum_{i=1}^{n} [-\ln p_i x + \theta\,(\ln x)^2 - \ln p_i(2-x) + \theta \ln x(2-x) + \theta \ln x(2-x) + \theta \ln(2-x)^2] = 0$$

$$\sum_{i=1}^{n} \theta[lnx^2 + 2lnx + 2ln(2-x) + \ln(2-x)^2] - [2lnp_i - lnx - \ln(2-x)] = 0$$

After making some mathematical adjustments, the following estimate is produced:

$$\hat{\theta} = \frac{\sum_{i=0}^{n} \mathrm{lnp_i lnx} - \mathrm{lnp_i}\ln(2-\mathrm{x})}{\sum_{i=0}^{n}[\mathrm{lnx}^2 + 2\mathrm{lnxln}(2-\mathrm{x}) + \ln(2-\mathrm{x})^2]} \qquad (9)$$

3. Result and Discusion

Application

The availability of data on a specific phenomenon is very important so that we can study it and reach certain results. However, in the case of being unable to obtain data from the relevant entity related to the study topic or if it is not sufficiently available, we can resort to another method, which is the simulation method (Ingalls, 2011), to obtain the necessary data to study that phenomenon. The simulation process is characterized by flexibility: it provides the ability to experiment and test by repeatedly running the simulation multiple times, changing the inputs of the estimation process each time. The importance of simulation comes from generating random numbers in the first experiment, which are independent of the random numbers in the subsequent experiment, and so on. Simulation can be defined as "the imitation or representation of reality using certain models (Müller & Pfahl, 2008).

Three different sample sizes were selected, which are N = 25, 50, and 75, where N = 25 represents small samples, N = 50 represents medium samples, and N = 75 represents large samples, with a hypothetical value chosen for the shape parameter

$\theta = 3$.

The results were compared using the Integral Mean Squared Error (IMSE) and the Integral Mean Absolute Percentage Error (IMAPE)

The following results were obtained:

Table 1 represents the reliability function values and the mean squared integrated error for different estimation methods in the third experiment according to the assumed sample sizes and at the default values of the Topp-Leone distribution parameters when the value of (θ=3) and with a sample size of (25).

The first model when (θ=3)						
mse per	**per**	**mse mle**	**mle**	**real**	**t**	**n**
0.000005	0.973573	0.0001509	0.9836084	0.9713258	0.8178407	25
0.0000055	0.9718117	0.0001666	0.982367	0.9694591	0.8056611	

The first model when (θ=3)						
mse per	**per**	**mse mle**	**mle**	**real**	**t**	**n**
0.0000455	0.8388181	0.0016915	0.8732011	0.8320729	0.4467696	
0.000049	0.8240797	0.0018428	0.8600057	0.8170775	0.6644011	
0.0000523	0.8094199	0.0019843	0.8467355	0.8021902	0.3539822	
0.0000599	0.7678096	0.0023323	0.8083616	0.760068	0.1708919	
0.0000691	0.6741481	0.0028058	0.7188065	0.6658369	0.6936687	
0.0000652	0.5369891	0.002772	0.5815611	0.5289116	0.545811	
0.0000644	0.5281054	0.0027445	0.5724658	0.5200779	0.5340936	
0.0000635	0.5180653	0.0027103	0.5621594	0.5100986	0.8188235	
0.000048	0.744282	0.0019201	0.7789272	0.7377118	0.5851943	

With a sample size of 25, it was found that the PER method is the best among the remaining methods in estimating the reliability function for the Topp-Leone distribution with the lowest mean integrated squared error for all experimental times, as its mean was 0.0000480 at a mean estimated reliability of the distribution of 0.7377118.

Table 2 represents the reliability function values and the mean squared integrated error for different estimation methods in the third experiment according to the assumed sample sizes and the default values of the parameters of the Topp-Leone distribution when the value of (θ=3) and with a sample size of (50).

The first model when (θ=3)						
mse per	**per**	**mse mle**	**mle**	**real**	**t**	**n**
0.0000338	0.9786957	0.0000265	0.9793617	0.9845129	0.2193951	
0.0001347	0.9459633	0.0001065	0.9472489	0.9575689	0.6539005	
0.0001771	0.9340162	0.0001403	0.9354797	0.947323	0.1930229	
0.0002061	0.9260248	0.0001634	0.9275973	0.9403811	0.2351118	
0.0002097	0.9250487	0.0001663	0.926634	0.9395288	0.558406	
0.0002555	0.9125693	0.0002029	0.9143098	0.9285545	0.2645025	50
0.0003051	0.8990632	0.0002426	0.9009552	0.9165302	0.7456089	
0.0003583	0.8843403	0.0002852	0.8863807	0.90327	0.7451588	
0.0003952	0.8739035	0.0003149	0.8760396	0.8937837	0.6795193	
0.0004934	0.844616	0.0003938	0.8469848	0.866829	0.3152886	
0.0002569	0.9124241	0.0002042	0.9140991	0.9278282	0.4609914	

With a sample size of 50, it was found that the MLE method is the best among the remaining methods for estimating the reliability function of the Topp-Leone distribution with the lowest mean integrated squared error for all experiment times, as its average was 0.0002042 with a mean estimated reliability of the distribution of 0.9278282.

Table 3 represents the reliability function values and the mean squared error integral for the different estimation methods of the third experiment according to the assumed sample sizes and at the default values of the Topp-Leone distribution parameters when the value of (θ=3) and with a sample size of (75).

The first model when (θ=3)						
mse per	**per**	**mse mle**	**mle**	**real**	**t**	**n**
0.0000005	0.9987485	0.000007	0.9996181	0.9994776	0.7573047	
0.0001045	0.972795	0.000007	0.9856571	0.9830171	0.7103637	
0.0002445	0.9523714	0.0000181	0.9722633	0.968009	0.3299003	
0.0004234	0.92989	0.0000337	0.9562692	0.9504674	0.7802228	
0.0004799	0.9231199	0.0000388	0.9512548	0.9450259	0.9417985	
0.0007707	0.8887562	0.0000667	0.9246832	0.9165172	0.5541976	75
0.0008342	0.8811672	0.0000731	0.9185966	0.9100493	0.799589	
0.001038	0.8560765	0.0000941	0.8979969	0.8882943	0.5972398	
0.0011187	0.8456941	0.0001028	0.8892772	0.8791404	0.3194725	
0.0011378	0.8431854	0.0001048	0.8871543	0.8769163	0.3555388	
0.0006152	0.9091804	0.0000539	0.9382771	0.9316914	0.6145628	

With a sample size of 75, it was found that the MLE method is the best among the remaining methods for estimating the reliability function of the Type I distribution with the lowest mean integrated squared error for all experimental times, as its average was 0.0000539 with an estimated reliability mean of the distribution of 0.9316914.

4. Conclusion

Based on the results obtained in this study, it can be concluded that the performance of the estimators improves as the sample size increases. Among the estimation methods considered, the maximum likelihood estimator demonstrates the best performance by producing the lowest mean squared error values. This indicates that the maximum likelihood method provides more accurate and reliable parameter estimates compared to the other methods analyzed. Therefore, the maximum likelihood approach is recommended for parameter estimation due to its superior estimation performance. In addition, future studies are encouraged to explore the Bayesian estimation method and apply it to real-world datasets in order to further evaluate its effectiveness and practical applicability.

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